DIFFERENCE BETWEEN KÄHLER CALCULUS AND GEOMETRIC CALCULUS

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I am often asked what is the difference between Kähler calculus (KC)[1] and geometric calculus (GC)[2]. Let me start with a few clarifications.

In Kähler's papers on this subject (1960-62), he dealt with tensor valued differential forms, with emphasis on scalar-valuedness. He provided very important applications for the latter, but none for the former, other than writing a very weird Dirac-type equation for them. For the comparison of KC and GC, we shall only need to deal with the scalar valued case, occasionally making brief incursions into anything more general. We shall thus use the abbreviation KC to refer to that case, unless we indicate differently.

In the main stream of mathematics, differential forms are meant to be antisymmetric multilinear functions of vectors (I follow Hestenes and deal with them through inner products in the tangent algebra, so that one does not need a second algebra for them). More importantly, Kähler's differential forms are not those ones, but rather are functions of r-surfaces. They constitute their own algebra. There is isomorphism to the tangent Clifford algebra, but not where differentiations are concerned. Whereas differentiation in the tangent algebra depends on the affine connection, differentiation in the KC algebra (for scalar valuedness!) does not, even though it looks so because of the appearance of Christoffel symbols. But these emerge here through the metric, not through the connection.

Kähler makes heavy use of components, which, in his case, have three series of indices. Two of them are for subscripts. One of these two is for multilinear functions of vectors, which, as we have seen, is not essential. The other series of subscripts is for differential forms. So, his tensor-valued differential forms can be seen as pertaining to the tensor product of two algebras: the tangent one and the Kähler one, i.e. of differential forms.

Since we are not interested here in tangent valuedness, we do not need connections. But, if we did, we could not avoid having to do with the product of algebras. Here is why: differentiation of a scalar-valued differential 1-form yields a differential 2-form, which is in the same algebra as what has been differentiated. On the other hand, differentiation of vector fields does not yield bivector fields, but vector-valued differential 1-forms.

The difference between the two calculi and the nature of the elements of the algebras that underlie those calculi are very consequential for their application to quantum mechanics. In a near future, I shall start documenting how superior Kähler's version is over any other version of relativistic quantum mechanics. Just try to reproduce with your favourite calculus what I shall achieve with the Kähler calculus.

References

[1] Kähker, E., "Der Innere Differentialkalkül", *Rendiconti di Matematica* **21** (1962) 425-523.
[2] Hestenes, D. and Sobczyk, G., *Clifford Algebra and Geometric Calculus*, Reidel (Boston, 1984).