KÄHLER'S QUANTUM MECHANICS

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Abstract

This is a paper in progress that is being written to motivate interest in the Kähler calculus (KC). It started as a note on its use for the foundations of quantum mechanics, where this calculus has so much to offer.

In this first incomplete version of the paper, we introduce Kähler's Dirac-like equation. It is not about spinors, particles or the electromagnetic interaction. These are concepts that emerge in the development of the equation for different applications.

Though it is actually an amplitude, the wave function of the KC carries more flavor of density matrices than of proper functions of operators. It is incompatible with the Copenhagen interpretation being a fundamental tenet of quantum mechanics. It is simply an emerging concept in specific circumstances. The interpretation that results from its conservation law is consistent with the experimental view of Carver Mead of electrons and, in general, particles as extended structures.

The energy of antiparticles does not emerge with a sign opposite to that of particles. There is no need for the concept of an (almost filled) infinite Dirac ocean of negative energy solutions.

We then briefly announce the connections of this incipient Kähler theory with several aspects of superconductivity theory and topological phase transitions. The forms of London's and of Landau-Ginzburg equations give credence to further development of the Kähler calculus for a new vision of the physics that he tried three decades after the formulation of that calculus through a "universal wave equation".

1 Introduction

In this paper, I intend to give a bird's eye view of the extraordinary power of the Kähler calculus (KC), published in 1960-62 [1],[2],[3]. I shall concentrate on its use in quantum mechanics, where its impact is greater.

A reader not familiar with this mathematics might ab initio consider preposterous some of the ideas proposed here. His mind should first be opened and his interest motivated. This is why this paper is what it is. But do not think that there is not a very serious mathematics behind the highlights presented here. Look at the original Kähler papers or at the chapters that I wrote for the Alterman 2016 event, which are reproduced in this web site. On the other hand and in case you were very impatient, I have made large use of boldface to give you a summary of this bird side view.

Consider the following interesting fact. As the notable French mathematician Bourguignon has pointed out, a Kähler paper of 1932 [4] that made him very famous (even more than some even more important work he did on the theory of exterior systems) not only constitutes important work on differential geometry, but has also impacted complex analysis, algebraic geometry, global analysis and theoretical physics (Key words for this work are Kähler metrics and Kähler manifolds).

Assume that a mathematician also trained as a forensic investigator has read, as you just have done, about the relevance of that 1932 work. Assume further that he has been made aware of the existence of his 1992 paper [5], called "Space-time Individual" (*Raum-Zeit Individuum*), and of its very ambitious goal: Kähler tried to show how to address what is "individual" from a universal perspective. His use of this term is not very clearly defined. But an implicit definition seems to be contained in his statement of being searching for a wave equation that will be valid for all quantum individuals. The paper intends to be an application of his KC (Just in case, I am simply reporting, not advocating this application he made of his calculus.

Our forensic mathematician might wonder why, in the last decade of his long life, Kähler returned to his 1960-62 and not to his famous 1932 work. I submit that he might have considered his calculus as one of his most important contribution to mathematics, possibly the most important one. Old work to which authors of such a stature return at a very old age suggest that they might consider it to be the most important part of their heritage, and to which any new ideas coming at a late time should be incorporated before "it becomes too late".

Assume finally that Kähler had written in English, from the Department of Physics of a top American university and with a degree of bluntness that is not characteristic of mathematicians when they speak of physics. He might have made pronouncements such as that the Copenhagen interpretation (as a basic tenet of quantum mechanics) and the emergence of particles and antiparticles with opposite signs are spurious effects of a deficient mathematics. Such pronouncements would have been paradigm changing. Fortunately for the paradigm, if such pronouncements were correct, the concomitant changes would occur everywhere in theoretical physics, but, in such a way, that, in some sense, almost everything remains close to what it presently is, or where it seems to be getting.

With that perspective in mind, I have taken the liberty of saying things that he stopped very short from saying. Not disclosing this would clearly amount to virtual plagiarism on my part. Starting on section 4, you can blame me exclusively for any connections I make between Kähler calculus and anything else.

2 The foundations of Kähler's Quantum Mechanics

2.1 The Kähler equation.

Kähler's quantum mechanics is an extensive application of an elegant equation of the KC that generalizes $y' = f(x)y$. It is called the Kähler equation (KEq). The one for tensor-valued differential forms can be found in [3]. But we are only interested in the one for scalar-valued differential forms [1], [2]. It reads :

$$\partial u = au, \tag{1}$$

where ∂ is a Dirac-type derivative operator and where juxtaposition of the differential forms a and u (respectively input and output in the equation) means their Clifford product. **The "wave function", u, is meant to be an element of the Clifford algebra of scalar-valued differential forms, and not** necessarily, much less primordially, a member of an ideal in this algebra, i.e. **a spinor** (These are very, very important in Kähler's work, but in context).

Assume that we replace a with the electromagnetic coupling (including the particle mass term) and we let u be a spinor. We then obtain an equation very much like the Dirac equation. But there would still be important differences, like the fact that u need not be a spinor, and not a fermion, and we are not presented with pairs of particle and antiparticles with energies of opposite sign. If one lets the math speak, a quantum mechanics without mysticism jumps from the equation without any preconceived philosophical ideas about the world.

The Clifford algebra that underlies the KC is not a tangent Clifford algebra, i.e. of the usual type, but of differential forms, understood as functions of hypersurfaces. It is built upon a module of differential $1-$forms, rather than upon a Euclidean or pseudo-Euclidean tangent vector space. It is defined by the equation

$$dx^{\mu} \vee dx^{\nu} + dx^{\nu} \vee dx^{\mu} = 2g^{\mu\nu}, \tag{2}$$

which is a disguised form of the "dot product",

$$dx^{\mu} \cdot dx^{\nu} = g^{\mu\nu}. \tag{3}$$

The relation between the last two equations should be obvious to practitioners of Clifford algebra, which is the main audience that we presently target. We shall refer to the algebra so defined as Kähler algebra, but be aware that he also dealt with something more general about which we shall speak in the next paragraph.

Whether expert or not, you may find these equations kind of strange. YES!, they are. Congratulations for becoming aware of this. I got the same impression when I first encountered them. Retrospectively, it simply happens that Eq. (2) (equivalently (3)) defines just a Clifford algebra of differential forms of scalar valuedness, the Kähler algebra. Progress is in the product of this algebra with its dual tangent Clifford algebra (blame me for this anticipation, which I have started to develop in papers posted in the arXiv). Kähler also built a more general algebra as tensor product of his scalar-valued one with the tangent tensor algebra built upon the tangent space dual to the module of differential $1-$forms. He developed scalar and tensor valuednesses in parallel. He did nothing with the last one other than writing for it a very weird Dirac-like equation. Here, we shall our hands full with Kähler algebra, sufficient for his magnificent quantum mechanical results. They would have been more than enough to earn him the Nobel Prize of Physics if he had done his work in the 1920's.

Not being a physicist, Kähler barely spoke of the enormous physical implications of his work. Thus, the conservation law that he obtained from (1) through the first Green identity of his calculus **is inimical to the Copenhagen interpretation**, since it concerns a magnitude that comprises two densities and corresponding currents in the same continuity equation. One of the densities is always non-negative and the other one is always non-positive. u is not, therefore, a probability amplitude, like a spinor in Dirac's theory is. Of course, one can always add the two densities and the two currents and recover the standard form. but in so doing one is losing information concerning this most important issue in the interpretation of quantum mechanics. We shall return to this in subsection 2.3 and 3.4. **The point here is not whether the Copenhagen interpretation is right or wrong**, but of when it applies and, specially, how close is or is not to "first principles". There is not even room here for the Einstein-Bohr debates. As we are about to see it is a matter of its place and role in Kähler's quantum mechanics.

The alternative interpretation of the wave function to be developed here is compatible with having electrons and any other purported point particle as extended particles. Others, like the towering experimental physicist Carver Mead [6], have already said it.

Kähler's **antiparticles appear with the same sign as particles**. **There is no need for an (**almost full) **infinite sea of negative energy solutions.** And, as this author has discovered in recent months, **Cooper pairs, and superconductivity and topological phase transitions fall nicely in the structure of his calculus**. And, if we go into Clifford valuedness, there are additional consequences from an "equation of everything" close to Eq. 1 that Kähler tried in 1992. The path to unification of classical and quantum physics emerges. Clifford valuedness would take us to the zoo of elementary particles.

2.2 Constant idempotents, spinor solutions and density matrices

Kähler showed the importance of the appropriate treatment of solutions with symmetry. But even the absence of symmetry may be relevant through expansion in terms of such solutions. This is not new. What is new is his bringing the study of symmetry-related quantum issues to a higher level of sophistication.

We should know that any differential equation or system of differential equations, whether ordinary or partial, can be written as an exterior system. A good example is the writing of Maxwell's equations with differential forms.

Let me make you aware of a qualified opinion expressed to me two decades ago at a differential geometry meeting of the Eastern United States by an expert on differential equations. In response to a question I asked him to know about what was going in his field, he said: the Cartan-Kähler theory is half of the theory of differential equations; the other half is everything else.

In general, solutions u are not members of ideals, much less are they particle states. The key of the relation between solutions in the algebra and solutions in ideals resides in the following. Idempotents define ideals. But if these are of a type called constant idempotents (denote them $Id^{\pm}$), the equation

$$\partial(uI^{\pm}) = (\partial u)I^{\pm}. \tag{4}$$

follows. We have not yet introduced here concepts needed to define constant idempotents. Suffice for the moment to assume that they indeed make equation (4) valid. From (1) and (4). One then gets

$$\partial(uI^{\pm}) = (\partial u)I^{\pm} = a(uI^{\pm}). \tag{5}$$

The last parenthesis is for emphasis. So, spinor solutions have been generated from solutions in the algebra. Remark: Particles are spinors, but one still needs more structure before one can say "this is a particle". Specific particles have to do with specific idempotents and, therefore, with specific spinors.

As we make progress in understanding what it takes for a spinor solution to become a particle, it will dawn on us that the concept of background depends on the answer given to "background in regard to what". The concept depends on the specific spinorial decomposition that one may be dealing with.

We shall later see **in connection with this emergence of spinor solutions** that **there is an intrinsic indeterminacy in Eq. (1)**. It will become obvious by contrast between the particle and non-particle contents of u, that the last one is a background field with a rather chaotic contents. It nevertheless obeys the same basic equation as the particles that live in it, and of which we shall speak later. We wish to advance that, in 1992, in *Raum-Zeit Individuum*, Kähler came back to an equation of type (1), assigning to it a rather universal generality. Again, what drives the argument of that work is not so much the form of the equation, but the specific idempotent decomposition that one selects.

The intrinsic indeterminacy in u is similar to that of the density matrix of standard quantum mechanics, **except that it is an amplitude**, not a density. Spinors, say those that represent well defined particles or even linear combinations of particles (but in general, they do not) emerge from u as islands of higher structure in a chaotic landscape. One cannot scape the *practical* indeterminacy of the background even if the equations are deterministic. But its coexistence with those structured solutions is consistent (to put it mildly) with the property of **Josephson** junctions to measure with greater precision that the background would seem to allow. The exchange **junctions have to be seen as an interchanging between two islands of structure. Each of the actors will change as if the unstructured background did not exist**, unless it becomes overwhelming and destroys the spinors.

In order to emphasize that, **in the KC, concepts that we associate with particles are already present in non-particle field solutions**, we shall not speak about particles for a few next pages. The association of those concepts with non-particles will be evident because the concepts emerge even before the particles do. The most striking example of this will be charge, but not in quantized form until particles "emerge". In the next subsection and section, we start to see how this can be, as one has to replace the probabilistic interpretation of the wave function with something else.

2.3 Dismissal of the Copenhagen interpretation

In Kähler's quantum mechanics, u also is an amplitude. We mean that densities and currents also are given by something similar to the bracket, $\langle \_ \mid \_ \rangle$, of Dirac's theory. This has to do with the form of the first Green identity in the KC. From the fact that the KEq lives in the algebra, and not exclusively in some ideal, the conservation law that pertains to the Kähler "wave function" (meaning wave differential form u) satisfies a different type of continuity equation. Translated to the vector calculus for better understanding, it reads as

$$\frac{\partial \rho_1}{\partial t} + div\ \mathbf{j}_1 + \frac{\partial \rho_2}{\partial t} + div\ \mathbf{j}_2 = 0, \tag{6}$$

where ρ_1 is nowhere negative and ρ_2 is nowhere positive. $\mathbf{j}_1$ and $\mathbf{j}_2$ are the corresponding currents. ρ_1 and ρ_2 (similarly $\mathbf{j}_1$ and $\mathbf{j}_2$) are determined respectively by ${}^{+}u$ and ${}^{-}u$, defined by the constant idempotents $\epsilon^{\pm}$, where

$$\epsilon^{\pm} = \frac{1}{2}(1 \mp i dt), \tag{7}$$

through the equation

$$u = {}^{+}u\,\epsilon^{+} + {}^{-}u\,\epsilon^{-}. \tag{8}$$

for more details, go to subsection 3.4.

The time t should be taken not as ordinary time, but as the closely related quantum mechanical time. We have denoted it as τ in our most recent papers posted in the arXiv.

Using the "signature" $(1, -1, -1, -1)$ instead of the one used by Kähler, $(-1, 1, 1, 1)$, removes the imaginary unit from (6) but not from everywhere where it appears in his work. Also, when you remove it from some places it reappears somewhere else. Also, Kähler theory can be greatly developed if one replaces the imaginary unit with elements of square minus one in real algebra. But Kähler did not raise any such issue, so we shall not do so until a later section.

Equation (7) is just a first decomposition of elements of the algebra over ideals. See for instance equations (11). There will be several more in due time. One of the great virtues of the KC is that idempotents can play a major role in physics. And there is a whole host of issues around them. For instance: do the ones commute with new ones that we may introduce? Do they contain $\epsilon^{\pm}$?, etc. In addition, options for new idempotents emerge if, instead of just using the algebra defined by (2), we use a tangent-valued algebra, etc.

Of course, it is easy to imagine that there will be situations or systems where the probability interpretation applies. The point is not that there is no use for **the Copenhagen interpretation** in Kähler theory. It **is not wrong, but an emergent or derived interpretation, rather than a basic tenet of the foundations of quantum physics**. It was adopted in the conceptual fog that accompanied the birth of quantum mechanics. It has become "truth" because virtually everybody repeats it, almost without question. If one expresses a different view, one does not belong, unless one is recognized to be in a position of too high authority to be criticized.

Scientists from different persuasion may have different reasons not to like or even dismissed it. In this calculus, the argument is very simple: its limitations and thus its not fundamental character are a matter of just computing without the need to invoke any physics. In fact, physical implications follow from the computations. A little bit more detail on this issue will follow in subsection 3.4

The superseding interpretation of u will emerge from its decomposition in terms of spinor solutions, and on whether these are particle states, superpositions of particle states or none of these. And one should be aware of the fact that one does not need to assume that a is the electromagnetic coupling, although this is the one with which we shall deal later on. Other couplings will simply give rise to different details in the expressions for (ρ_1, $\mathbf{j}_1$) and (ρ_2, $\mathbf{j}_2$) in terms of ^+u and ^-u respectively, or in terms of a different decomposition of the algebra into a sum of ideals. of great importance however is the fact that no terms in (5) mix the ^+u with the ^-u.

3 From fields to particles

3.1 Particle concepts in fields without particles

A particular input a can be $m + eA$, where A is a differential $1-$form and where m and e are constants. Solutions in the ideals will now satisfy equations

$$\partial(u\epsilon^{\pm}) = (m + eA)(u\epsilon^{\pm}). \tag{9}$$

(Other constants that this equation may raise in readers' minds are irrelevant for present purposes and thus ignored here).

One does not need to have a physical interpretation to m, e and A for developing equation 9. One still obtains the *form* of the Pauli-Dirac equation and, in the next approximation, the Foldy-Wouthuysen equation. And one does not need to resort to analogy with non-relativistic quantum mechanics for help. Concepts like the general momentum operator emerge; operator theory is not necessary for development of Kähler's quantum mechanics, as these operators come embedded in the computations, each in its own way.

Assume now that we assign the standard interpretation to m, e and A. **The conserved quantity** to which the previous section referred should now be identified with a general concept of **charge**. We said "general" because it **is defined and can be computed for any given u, regardless of whether it represents a particle or not**. It is a matter of integrating the ρ's. And if a is the electromagnetic coupling, ρ must obviously be identified with density of what we know as electric charge. If there are no particles in the field (a clear example would be if there were not enough energy to make a particle), it still is a valid concept, but not yet quantized.

Assume further that we specialize (9) to $A = -(edt/r)$ and solve the equation. We get the fine structure of the hydrogen atom. Nowhere in what we have described is there a need for Pauli or Dirac matrices. One obtains standard quantum-mechanical results for standard problems. The interpretation now is different. If the wave differential form is not a probability amplitude, it is to be thought as providing the details of the extension that Mead predicated for the electron.

Spinors, the Hilbert space scenario and the concept of probability amplitude are emerging concepts, thus not belonging to the foundations of quantum physics. Several such concepts emerge even before we introduce the concept of particle (see below), as has been the case with charge (see above). Different concepts emerge directly from different equations, like (1), (5), (6), (9) so far. **The emergence of electrons and positrons from the KC will later be seen as being one and the same with the quantization of charge**.

3.2 The pairs (rest mass, charge) and (angular momentum, chirality)

Kähler theory reaches the standard conclusion of characterizing particles by their mass and spin, but in a different way from group representation theory. It is a matter of judiciously choosing idempotents and, among these, the primitive ones. **Charge is related to energy by virtue of their common relation to time translations, respectively through dt and** t (Present day physics has failed to realize this because it has failed to look at differential equations as per the Cartan-Kähler theory of exterior systems). Very interesting theory results from multiplication of $\epsilon^{\pm}$ with other constant idempotents with which they commute. This excludes the other translation idempotents, as they do not commute with $\epsilon^{\pm}$.

The role of $\partial/\partial t$ is now played by $\partial/\partial\phi$, not by a **replacement of r and p in r $\times$ p with operators**, which **is an unnecessary particle-related approach**. The idempotents associated with rotations are

$$\tau^{\pm} = \frac{1}{2}(1 \pm i dx dy). \tag{10}$$

This may look a little bit strange because one might have expected this expression to involve $d\phi$. It is actually buried in $dxdy$. To make the argument short, let us simply mention that $dxdy$ is a constant idempotent, but $d\phi$ is not. The two signs inside the parenthesis correspond to two different chiralities in an obvious way. **Spin and chirality have to do with the dependence of the wave function on ϕ and $d\phi$ respectively**.

The $\tau^{\pm}$'s annul each other and commute with the $\epsilon^{\pm}$, which also annul each other. As a result of all of this, the idempotents $\epsilon^{\pm}\tau^{*}$, where τ^{*} stands for both $\tau^{\pm}$ and $\tau^{\mp}$, also annul each other.

We can use the four $\epsilon^{\pm}\tau^{*}$ to replace $(1, dt, dxdy, dtdxdy)$ in u, so that we get any member of the algebra as a sum of elements

$$u = \ {}^{+}u^{+}\ \epsilon^{+}\tau^{+} + \ {}^{+}u^{-}\ \epsilon^{+}\tau^{-} + \ {}^{-}u^{+}\ \epsilon^{-}\tau^{+} + \ {}^{-}u^{-}\ \epsilon^{-}\tau^{-}. \tag{11}$$

The coefficients of these idempotents can be computed from the fact that the equation

$$1 = \epsilon^{+}\tau^{+} + \ \epsilon^{+}\tau^{-} + \ \epsilon^{-}\tau^{+} + \epsilon^{-}\tau^{-}. \tag{12}$$

yields

$$u = u\epsilon^{+}\tau^{+} + u\epsilon^{+}\tau^{-} + u\epsilon^{-}\tau^{+} + u\epsilon^{-}\tau^{-} \tag{13}$$

through multiplication by u on the left. Because of mutual annulment of the idempotents, multiplication of (13) by $\epsilon^{\pm}\tau^{*}$ on the right yields

$$u\epsilon^{\pm}\tau^{*} = \ {}^{\pm}u^{*}\ \epsilon^{\pm}\tau^{*}. \tag{14}$$

There is a great difference between (11) and (13). The ${}^{\pm}u^{*}$ do not contain the $(1, dt, dxdy, dtdxdy)$. They are computed through (14), thanks to the properties of the idempotents involved.

The four products $\epsilon^{\pm}\tau^{*}$ are primitive idempotents. Readers who are not familiar with this concept need only know that one cannot carry further multiplying $\epsilon^{\pm}\tau^{*}$ with other idempotents and still get results such as (14). The reason for this impossibility (backed by a Radon-Hurwitz theorem and also by trial until exhausting the options) is the lack of commutativity of $\epsilon^{\pm}\tau^{*}$ with additional idempotents factors.

3.3 Electrons and positrons, but not as point particles and not with opposite energies

A Dirac spinor can be thought of as being the first column of a 4×4 matrix, the other columns then being made of zeros. The four entries in the column accommodate respective right and left handed positrons and electrons, but with different sign of the energy as is well known. If matrix representations were used in Kähler's theory, the role taken by the four elements in the spinor would be taken by the four columns in the four by four matrix, but all of them corresponding to the same value, sign included, of the energy if the matrix represented a stationary solution. But matrix representations are not needed (column matrices in particular) and simply create confusion.

In Kähler's quantum physics, spinors are simply members of ideals into which Clifford algebras can be decomposed, as we have just seen. But being a spinor in an ideal defined by a constant idempotent is not enough to represent a particle. Kähler clearly saw this when he proposed the formulas

$$e^{im\phi}e^{-iEt/\hbar}\ f(\rho, z, d\rho, dz)\epsilon^{\pm}\tau^{*} \tag{15}$$

for electrons and positrons with both chiralities (speaking of chiralities is more appropriate than speaking of spin, which will become increasingly apparent in future sections). **Antiparticles have to do with the idempotent that enters the spinor in the wave function for a given particle. What makes the positron a positron is its pertaining to the left ideal defined by ϵ^{+}, not negative energies**.

In principle, there could be negative sign of energy for both electrons and positrons, as is the case for m. But this is a separate issue. The KC does not speak of why the negative values of energy are not needed, which might have to do with the arrow of time or whatever. But it speaks of the fact that one does not need opposite energies to have particles and antiparticles.

Notice that there is not an $\hbar$ in the first exponential in (15). This has to do with the way in which Kähler defined angular momentum, without direct or indirect resort to the concept of angular momentum operator in classical physics. It is viewed simply as $\partial/\partial\phi$, where ϕ plays the role of azimuthal coordinate. Of course, his m is dimensionless. We shall return to this factor in the next section, in connection with a paper by Kosterlitz dealing with a 2-D lattice of spins.

For Mead, an electron has the property of adapting to its environment, be it a hydrogen atom or a wire. He claims that experiments are regularly performed with neutrons that are one foot across, and that, in his laboratory, he can make electrons that are ten feet long. He further makes statements like "The electron ... is the thing that is wiggling, and the wave is the electron". His use of the term wave is not the standard one of classical electrodynamics, to which he would refer as non-coherent electrodynamics. He simply means a solution of a quantum mechanical wave equation. This remark about his use of electrons as waves is crucial in order not to misunderstand him in what follows.

Mead argues that the experimental support for the point particle view was due to experimental technique that was too primitive when such a concept was adopted. In accordance with this view and what we have learned from Kähler about what the concept of an electron is, the factor $f(\rho, z, d\rho, dz)$ **represents the shape that the extended electron or positron takes as a function of what confines it, or what it is attached to.**

One can always decompose a spacetime differential form into elements of the four algebras defined through multiplication by $\epsilon^{\pm}\tau^{*}$ on the right. But it is a very major requirement that the coefficients will be $e^{im\phi}e^{-iEt/\hbar}$ $f(\rho, z, d\rho, dz)$ for any of the components. In general, one will have factors $f(t, \rho, \phi, z, d\rho, dz)$ for elements in those ideals. As a prelude to the role that will later be assigned to all sorts of idempotents, let us say that comparable statements can be made for macroscopic coherent states, like superconductors. The idempotents and thus the ideals and the factors f will be different from those for electrons and positrons, while still having very constrained forms, i.e. they would be highly structured. Transactions between superconductors in almost contact can only occur in "amounts" such that the structured elements in the transaction retain their very specific special forms. Changes in a highly structured member of an ideal must in general be compensated by a change in another similar structure that makes part of the same system, like, say, in a Josephson junction. The mathematics allows for changes that occur in amounts that are much smaller than internal fluctuations of the system comprising both spinors (respectively alternative structures). But, in principle, the theory does not speak at this point about a change in an structured element in an ideal also involving a non-structured ideal in the same decomposition of u.

3.4 The quantization of charge

The interpretation of the wave function as an amplitude of charge raises an issue of normalization different from the same issue under the Copenhagen interpretation, here inimical. The emergence of electrons and positrons as structures from an electromagnetic substratum and the fact that they have identical wave functions (except for a sign in a exponent) leads one to assign the value of minus one to the charge of the electron. That is the natural normalization in this case. Notice that it does not depend on the unit chosen for any other magnitude.

For this proposed normalization to be justified, we have to make sure that the concept so introduced is consistent with the experimental data, namely that, if we have a system of n electrons, its charge is $-n$. In more detail, the continuity equation of which we spoke in the previous section is

$$[u, u] = \frac{1}{2}\{^{+}u, ^{+}u\} + \frac{1}{2}\{^{+}u, \eta^{+}u\}_1 \wedge icdt - \frac{1}{2}\{^{-}u, ^{-}u\} + \frac{1}{2}\{^{-}u, \eta^{-}u\}_1 \wedge icdt. \quad (16)$$

At this point, we do not need to understand the details of this equation, which would take too long. It suffices to know that there is no term that mixes ^{+}u with ^{-}u.

Assume next that we have an ensemble of electrons. If the background is neglected, the wave function will be of the form

$$^{-}u = \sum_{i=1}^{n} {}^{-}u_i = {}^{-}u_0 \sum_{i=1}^{n} \delta(\mathbf{r} - \mathbf{r}_i), \quad (17)$$

where $\mathbf{r}_i$ is the position of the center of each electron and where $^{-}u_0$ is given by Eq. (15). The density of charge is quadratic in u_0. When integrating it over a volume and because of the the delta functions, the integral will become a sum over integrals, one for each electron. In other words, the sum of the charge of the electrons will be the number of electrons preceded by the minus sign.

Formula (17) should include the electromagnetic background, i.e. the interaction. But this argument cannot be made more rigorous without invoking more detail about the wave function of the ensemble, which in actual practice amounts to the thermodynamic treatment of the issue, which is far beyond present development of the theory. Let us then simply remark that, except under extraordinary circumstances where we would not have electrons but a primordial soup, the dominant energy would be in the electrons as individuals, and so would be the closely elated charge. As we further develop Kähler's theory in later sections, the issue of charge will be more deeply understood. For instance, we shall understand that equations (16) and (17) and the Kähler equation itself is not for fermions, but also for bosons, not only for electrons and positrons, but also for Cooper pairs. So, let us say that we have at least made theoretically plausible that measurable charge in an ensemble of electrons and/or positrons will be the number of charges taking into account their respective signs.

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