CHAPTER 3: Kähler Differentiations

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This chapter will be used for the second day of the second phase of the summer school. Said better, this chapter together with some complementary work posted in the arXiv will be the source from which my collaborators will choose what can and should be taught in view of the available time. I shall try to leave others do the choice so that, in the end, the instructors of this subject will jointly decide on what participants will listen to.

These notes are longer than I intended them to be because they address, in addition, issues that readers with early access to them have raised. One recurring theme is that the concept of differential $r-$form does not quite sink. It is an $r-$integrand, thus a function of $r-$surfaces You evaluate the differential $r-$form by integrating it on the on the $r-$surface. Take $3xdx + dy$. This does not mean $3x\Delta x + \Delta y$ regardless of how small Δx and Δy may be. We rather have

$$\int_{x,\, y}^{x+\Delta x,\, y+\Delta y} 3xdx + dy = \left[\frac{3x^2}{2} + y\right]_{x,\, y}^{x+\Delta x,\, y+\Delta y} = 3x\Delta x + \frac{3}{2}(\Delta x)^2 + \Delta y.$$

It is at this point that, if and only if Δx is arbitrarily small we can validate the expression $3x\Delta x + \Delta y$. A differential form is nothing like an infinitesimal change or anything of the sort. The traditional interpretation of a differential form as an infinitesimal begs the definition of infinitesimal.

If there is anything that could be a little bit difficult to understand is what does it mean that $dx^2 = dy^2 = 1$ (also written as $dx \vee dx = dy \vee dy = 1$). But this is just a particular case of what does it mean that the dot product of a differential $r-$form and a differential $1-$form is a differential $(r-1)$-form? This question, specially in the particular case of $dx^2 = dy^2 = 1$ because it is so simple, is clamoring for something more sophisticated and to which we have referred in the past as the Cartan-Kähler calculus. But we are not yet ready for that!

We insist on a point made in the previous chapter, namely that we write $dx^2 = dy^2 = 1$ only if x and y are Cartesian coordinates. In general, the differential $1-$forms whose square is ± 1 are those which orthonormalize the metric when it is viewed as a quadratic differential form. This quadratic differential form is a $2-$tensor built upon a module of differential 1-forms by tensor product. It does not belong to the Kähler algebra, but one can have it in parallel to it. You would have to wait for a Cartan-Kähler calculus for a better course of action.

As a concession to those who cannot live with formulas like $dx^2 = dy^2 = 1$, consider the following example:

$$(du)^2 = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

You might wish to stop at ,

$$(du)^2 = \left(\frac{\partial u}{\partial x}\right)^2 dx^2 + \left(\frac{\partial u}{\partial y}\right)^2 dy^2 = \left(\frac{\partial u}{\partial x}\right)^2 dx \cdot dx + \left(\frac{\partial u}{\partial y}\right)^2 dy \cdot dy$$

and keep going until you would have to actually use it for something.

1 Differentiations

In Kähler's work, all differentiations other than partial differentiation are obtained from the so called covariant derivative. We refer readers to chapter one for the basic concepts at a deeper level than here, as we do not presently require as deep a knowledge of differential geometry as is the case there.

In this chapter, we shall deal only with scalar-valued differential forms. And we shall refer to greater valuedness only occasionally, for the purpose of achieving deep understanding of some important issue.

In order to provide early perspective to those who have already worked at a deep level with differential forms or Clifford analysis, we used the term divergence to what, from this point on, will be referred to as interior derivative. The reason is that the term divergence does not capture the richness and difference with other calculi of the Kähler calculus (KC). In this calculus, the interior derivative of a vector field is zero. So, it is not desirable to use the term divergence for the interior derivative. On the other hand, the interior derivative of a differential 1-form takes the form of the divergence of what is referred to as the divergence of a vector field in the standard literature. From the perspective of the KC, concepts that in the vector calculus are assigned to $r-$tensors should be attributed to differential $r-$forms. That frees the valuedness structures (built upon contravariant and covariant vector fields) for other purposes.

1.1 Kähler's approach to covariant differentiation

Readers may start this chapter by simply accepting formula (1.5) below, and looking at the text up to that formula only anecdotally. This amounts to accepting without proof Eq. (1.5) rather than doing so with equation (1.2). If they want to know better, they should read as far as they can the first chapter and then returning here.

Kähler represented tensor-valued differential forms with the notation

$$u^{j_1...j_p}_{i_1...i_p} = a^{j_1...j_q}_{i_1...i_p k_1...k_m} dx^{k_1} \wedge \ldots \wedge dx^{k_m}. \tag{1.1}$$

He then proceeded to introduce in ad hoc manner a concept of covariant differentiation of tensor-valued differential forms as follows:

$$\begin{aligned} d_h a^{k_1...k_\mu}_{i_1...i_\lambda\ l_1...l_p} = & \frac{\partial}{\partial x^h} a^{k_1...k_\mu}_{i_1...i_\lambda\ l_1...l_p} + \\ & + \Gamma^{k_1}_{hr} a^{r...k_\mu}_{i_1...i_\lambda\ l_1...l_p} + ... + \Gamma^{k_\mu}_{hr} a^{k_1...r}_{i_1...i_\lambda\ l_1...l_p} \\ & - \Gamma^{r}_{hi_1} a^{k_1...k_\mu}_{r...i_\lambda\ l_1...l_p} + ... + \Gamma^{r}_{hi_r} a^{k_1...k_\mu}_{i_1...r\ l_1...l_p} \\ & - \Gamma^{r}_{hl_1} a^{k_1...k_\mu}_{i_1...i_\lambda\ r...l_p} + ... + \Gamma^{r}_{hl_r} a^{k_1...k_\mu}_{i_1...i_\lambda\ l_1...r_p}, \end{aligned} \tag{1.2}$$

where the Γ's are the Christoffel symbols. There is here not only a matter of contents but also of notation. The use of components to such a large extent is reminiscent of the tensor calculus.

For several chapters, we shall be interested only in scalar-valued differential forms. Hence, we may ignore lines 2 and 3 and write

$$d_h a_{\ l_1...l_p} = \frac{\partial}{\partial x^h} a_{l_1...l_p} - \Gamma^{r}_{hl_1} a_{\ r...l_p} + ... + \Gamma^{r}_{hl_r} a_{\ l_1...r}. \tag{1.3}$$

Kähler invoked the equation

$$\omega^k_i = \Gamma^k_{ij} \cdot dx^j, \tag{1.4}$$

which may mean different things to those who are not familiar with equations of structure. He used (1.4) to rewrite (1.3) as

$$d_m u = \frac{\partial u}{\partial x^h} - \omega_m{}^r \wedge e_r u. \tag{1.5}$$

So, there is a term additional to the partial derivative. It is zero if we have Cartesian or pseudo-Cartesian (if you prefer to use this term when dealing with pseudo-Euclidean spaces). We shall barely need to use it in this course; we can do very much without it.

In deriving Eq. (1.5), Kähler paid the price of going against the natural order of things, namely the one that we proposed in chapter 1; the foregoing formula (1.4) should precede formula (1.3). More importantly, we showed that one does not need to derive (1.5), much less memorize this and many other formulas. We shall now show —only for those really interested— how (1.5) can be derived by resorting to the equations of structure of a manifold endowed with a metric but not with an affine structure. But even if we have an affine structure where the affine connection is not Levi-Civita's, there are issues for which the affine structure is not relevant, but the Christoffel symbols are. When the symbols Γ^k_{ij} are the Christoffel symbols (most of the time from now on), we should not forget that they were born half a century before they were seen as a tool for parallel transport, simultaneously by Levi-Civita, Hessenberg and Schouten.

1.2 Cartan's-like approach to covariant differentiation

Cartan did not deal explicitly with the issue of covariant differentiation. His exterior derivative of scalar-valued differential forms and of vector-valued differential forms are equivalent to what, in the modern literature, are known as exterior and as exterior covariant derivatives (or simply covariant derivatives for tensor fields). Something similar is the case with Kähler, although the latter paid more attention to tensor-valuedness and less attention to structure than Cartan. Let us then proceed like the latter would have done if they had explicitly considered covariant and interior derivative.

At its most general, a differential form u will be a sum of terms of the form $\alpha = a\ \omega^1 \wedge ... \wedge \omega^s \wedge ... \wedge \omega^m$ on some differentiable manifold of dimension $n \geq m$. Let us exterior differentiate a monomial $\alpha = a\ \omega^1 \wedge ... \wedge \omega^s \wedge ... \wedge \omega^m$, and then postulate the distributive property of d_h. Proceeding in this way, we recover the exterior calculus and the essence of a reinterpreted vector calculus.

Before differentiating each ω^i, we move it to the front and introduce alternating factors 1 and -1. So, with obvious simplification of notation, and with $1...\bar{s}...m$ meaning the absence of s,we have

$$d\alpha = da \wedge \omega^{1...m} + \sum_{s=1}^{s=m} a(-1)^{s-1} d\omega^s \wedge \omega^{1...\bar{s}...m} = da\ \omega^{1...m} + \sum_{s=1}^{s=m} d\omega^s \wedge e_s\alpha. \tag{1.6}$$

But

$$\sum_{s=1}^{s=m} d\omega^s \wedge e_s\alpha = d\omega^r \wedge e_r\alpha, \tag{1.7}$$

with summation from $r = 1$ to $r = n$ and not just to m. This is because $e_r\alpha$ is $\omega_r \cdot \alpha$ and $\omega_r \cdot \omega^i = \delta^i_r$. terms on the right where i is greater than m

do not contribute to the sum $\omega_r \cdot \alpha$. Equation (1.1) can thus be written further as

$$d\alpha = a_{/h}\ \omega^h \wedge \omega^{1...m} + d\omega^r \wedge e_r \alpha, \tag{1.8}$$

with $a_{/h}$ defined by $da = a_{/i}\ \omega^i$. We next use

$$d\omega^r = \omega^k \wedge \omega_k{}^r = -\Gamma_k{}^r{}_h \omega^h \wedge \omega^k = \omega^h \wedge (-\Gamma_k{}^r{}_h \omega^k). \tag{1.9}$$

Hence

$$d\alpha = \omega^h \wedge a_{/h}\ \omega^{1...m}\ -\ \omega^h \wedge (-\Gamma_k{}^r{}_h \omega^k) \wedge e_r \alpha, \tag{1.10}$$

We define the covariant derivative $d_h \alpha$ as

$$d_h \alpha = a_{/h}\ \omega^{1...m} - \Gamma_k{}^r{}_h \omega^k \wedge e_r \alpha. \tag{1.11}$$

Under the distributive property of differential operators with respect to the sum, we further have

$$d_h u = \frac{\partial u}{\partial x^h} - \omega_h{}^r \wedge e_r u, \tag{1.12}$$

for an arbitrary differential form u, whether of homogeneous grade or not. That is what Cartan would have done.

We are now interested in connecting with Kähler's work, i.e. in specializing this to coordinate bases. In general bases, the $\Gamma_k{}^r{}_h$ are defined by $\omega_k{}^r = \Gamma_k{}^r{}_h \omega^h$ and are not in general equal to the Christoffel symbols, which constitute the particular cases for coordinate bases. In terms of them, we have. In terms of coordinate bases

$$\Gamma_k{}^r{}_h = \Gamma_h{}^r{}_k \qquad ...\Gamma_k{}^r{}_h dx^k = \Gamma_h{}^r{}_k dx^k = \omega_h{}^r. \tag{1.13}$$

1.3 The Kähler derivative

The Kähler differential is defined as

$$\partial u = dx^h \vee d_h u = dx^h \wedge d_h u\ +\ dx^h \cdot d_h u = du + \delta u, \tag{1.14}$$

where

$$du = dx^h \wedge d_h u, \qquad \delta u = dx^h \cdot d_h u. \tag{1.15}$$

Clearly

$$\partial u = dx^h \vee \left(\frac{\partial u}{\partial x^h} - \omega_h{}^r \wedge e_r u \right) = dx^h \vee \frac{\partial u}{\partial x^h} - dx^h \vee (\omega_h{}^r \wedge e_r u). \tag{1.16}$$

We further have

$$\partial u = dx^h \vee \frac{\partial u}{\partial x^h}\ -\ dx^h \wedge \omega_h{}^r \wedge e_r u\ -\ dx^h \cdot (\omega_h{}^r \wedge e_r u). \tag{1.17}$$

Since $dx^h \wedge \omega_h{}^r$ equals zero, we finally have

$$\partial u = dx^h \vee \frac{\partial u}{\partial x^h}\ -\ dx^h \cdot (\omega_h{}^r \wedge e_r u) = dx^h \vee \frac{\partial u}{\partial x^h}\ -\ e^h (\omega_h{}^r \wedge e_r u). \tag{1.18}$$

1.4 Leibniz rules

You should be able to show that

$$d_h(u \vee v \vee w \vee ...) = d_h u \vee v \vee w \vee ... + u \vee d_h v \vee w \vee ... + u \vee v \vee d_h w \vee ... + ... \tag{1.19}$$

and that

$$d_h(u \wedge v \wedge w) = d_h u \wedge v \wedge w \vee ... + u \wedge d_h v \wedge w \vee ... + u \wedge v \wedge d_h w \vee .... + ... \tag{1.20}$$

We have used so many terms in order to emphasize the lack of alternation of positive and negative signs. The proofs, essentially the same in both cases, are obvious by showing that, at each point, we can make ${\omega_h}^r$ equal to zero. The justification of making this annulment is specially simple for the Christoffel symbols by resorting to the equation for the geodesics. It can also be achieved —in a different and less easy way— in more general cases, as per section 8.4 of my book. We then have the Leibniz rule for the partial derivative . Since the proof is valid at each point independently of what happens at other points, and since the result involves only invariant terms, the proof is complete.

An important rule to accompany the exterior differential of an exterior product of differential forms is the Kähler derivative of the Clifford product of differential forms. The rule is

$$\partial(u \vee v) = \partial u \vee v + \eta u \vee \partial v + 2e^h u \vee d_h v \tag{1.21}$$

The symbol η stands for changing every differential 1−form factor by its opposite. So $\eta\alpha_r$ equals α_r or $-\alpha_r$ depending on whether the grade of α_r is even or odd. The symbol e^h is defined by its action $e^h u = \omega^h \cdot u$.

We proceed to prove it. In terms of coordinate bases, the differentiation ∂u was introduced as

$$\partial u = dx^h \vee d_h u. \tag{1.22}$$

Hence

$$\partial(u \vee v) = dx^h \vee d_h(u \vee v) = dx^h \vee d_h u \vee v \ + \ dx^h \vee u \vee d_h v \tag{1.23}$$

In the last term, we want to move dx^h past u, for which purpose, we solve for $dx^h \vee u$ in

$$dx^h \vee u \ - \ \eta u \vee dx^h = 2dx^h \cdot u = 2e^h u \tag{1.24}$$

and replace it in (1.24). Equation (1.21) follows.

It is worth considering other "Leibniz", not so much because they are going to be used, but to illustrate that one has to be careful and not extrapolate unduly. Without proof:

$$d(u \vee v) = du \vee v \ + \ \eta u \vee dv \ + \ e^h u \vee d_h v \ - \ \eta d_h u \vee e^h v. \tag{1.25}$$

Notice the absence of exterior products.

Let us do an example. Let z be defined as $z = x + ydxdy$. Then

$$z^2 = x^2 - y^2 + 2xydxdy. \tag{1.26}$$

Its exterior derivative is

$$d(z^2) = 2xdx - 2ydy. \tag{1.27}$$

Assume now that we wish to apply the Leibniz rule that we have just derived. The sum of the first two terms can be written as $2dx \wedge (x + ydxdy)$ which is $2xdx$. For the last two terms (actually four when developed) we use

$$e^x z = ydy, \qquad e^y z = -ydx, \qquad d_x z = 1, \quad d_y z = dxdy,$$

where $e^x = e^1$ and $e^y = e^2$. We use these equations to get the sum of those four terms, which yield $ydy - ydy - ydy - ydy = -2ydy$. This checks the formula. It was not pretty. In this case we could first compute the product, so that we did not need to apply the rule, but there will be cases where there may be no alternative to use that "Leibniz rule".

A similar complex rule bearing structural similarity is

$$\partial(u \wedge v) = \partial u \wedge v \ + \ \eta u \wedge \partial \wedge v \ + \ e^h u \wedge d_h v \ + \ \eta d_h u \wedge e^h v. \tag{1.28}$$

In the derivations of the formulas for $d(u \vee v)$ and $\partial(u \wedge v)$, one simply resorts to the contractions of components of covariant derivatives with the elements of a basis of differential $1-$forms.

1.5 Examples of covariant and interior differentiations

We proceed to provide examples of interior differentiation of scalar-valued differential 1-forms. Under a different notation, you can find these formulas in his 1961 paper. I post them here since one is not used to hear or speak of covariant derivatives of the differentials of the coordinates. It is only a matter of applying the formulas that we have given. Instead of indices 1, 2, 3 and 4, I shall use ρ, ϕ, z and t.

The only non-null Christoffel symbols are

$$\Gamma_{\phi}{}^{\rho}{}_{\phi} = -\rho, \qquad \Gamma_{\rho}{}^{\phi}{}_{\phi} = -\frac{1}{\rho} = \Gamma_{\phi}{}^{\phi}{}_{\rho}.$$

The ω's that we require are

$$\omega_{\phi}{}^{\rho} = -\rho d\phi, \qquad \omega_{\rho}{}^{\phi} = \frac{d\phi}{\rho}, \qquad \omega_{\phi}{}^{\phi} = \frac{d\rho}{\rho}. \tag{1.29}$$

The covariant derivatives then are

$$d_{\rho}u = \frac{\partial u}{\partial \rho} - \frac{\partial \phi}{\partial \rho} \wedge e_{\phi}u, \qquad d_z u = \frac{\partial u}{\partial z}, \qquad d_t u = \frac{\partial u}{\partial t},$$

$$d_{\phi}u = \frac{\partial u}{\partial \phi} + \rho d\phi \wedge e_{\rho}u - \frac{d\rho}{\rho} \wedge e_{\phi}u. \tag{1.30}$$

Exercise. Find the divergence in cylindrical coordinates in 3-D Euclidean space. Needless to say that you could follow the same process for arbitrary coordinate systems.

We want to make an important clarification to preempt confusions. When one computes the interior derivative —in say cylindrical coordinates— of a differential $1-$form, one gets the same expression that one gets in the literature for the divergence of a vector field. We are not dealing here with tensor-valued differential forms, and with vector-valued ones in particular. If one used information given in chapter 1 or the beginning of chapter 2 to compute the interior derivative of a vector field, you would get zero. the reason is simple. The covariant derivative of a vector field in the sense of KK is another vector-valued $0-$form. Its interior contraction with differentials 1-forms is zero (you get scalars with interior contraction of differential $1-$forms with differential $1-$forms, not of scalars with differential $1-$forms, regardless of valuedness. Does it contradict the physics? No, in the KC, the so called magnetic vector field is a differential $2-$form, and the vector potential is a differential $1-$form. The divergence of the magnetic vector field is obtained as the exterior derivative of the magnetic differential $2-$form. In the next section, we shall understand this a little bit better in the next section when we deal with the concepts of Hodge duals and coderivative.

Consider now the metric. There are definitions for all tastes. The symmetric quadratic differential form $g_{ij}dx^i dx^j$ is a tensor product of differential 1-forms is a tensor product, $g_{ij}dx^i \otimes dx^j$ and does not belong to what we have called Kähler algebra, which is a Clifford algebra of scalar-valued differential forms, not a tensor algebra. In the KC, the metric tensor of which one says that its covariant derivative is $g^{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, equivalently $g_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$. In other words it is a 2-tensor-valued differential $0-$form.

2 Hodge duality, co-derivative and Laplacians

The term duality has several meanings in algebra, calculus and geometry. Given a vector space, one speaks of its dual vector space. Given an n-dimensional Euclidean space, one speaks of the $(n-1)$-plane dual of a point. Hodge duality is a third concept not too different from the previous one but more comprehensive. It is important because the concept of co-derivative in computationally less comprehensive calculi resorts to it to deal with situations which are better addressed with the interior derivative of the Kähler's calculus.

The "co-derivative version" of the interior derivative form allows one to understand how versions of the Laplacian which apparently have different numbers of terms are all part of the same general formula which takes simpler forms depending of the object to which it is being applied.

2.1 Hodge duality

Let us use the symbol z to refer to the unit differential form in the Kaehler algebra. It is undefined up to the sign, depending on the order in which the differential $1-$form factors are chosen. As defined before, let (ω^i) be a set of n differential $1-$forms that orthonormalize the metric. Let the symbol z denote the unit hypervolume, i.e. the differential $n-$form

$$z = \omega^1 \wedge \omega^2 \wedge ... \wedge \omega^n. \tag{2.1}$$

This is undefined by a sign depending on the order in which we pick the elements of the basis of differential $1-$forms. We shall choose the order so that this expression coincides with

$$z = |g_{ij}|^{1/2} \ dx^1 \wedge dx^2 \wedge ... \wedge dx^n. \tag{2.2}$$

If you were hesitating as to whether the exponent of $|g_{ij}|$ is $+1$ or -1, just check with the element of area in polar coordinates.

We define the Hodge dual (in the following, called simply the dual) $*u$ of a differential form u as

$$*u = u \vee z, \tag{2.3}$$

For positive definite signature, we have

$$** = zz = (-1)^{\binom{n}{2}}, \qquad z^{-1} = (-1)^{\binom{n}{2}} z, \qquad *^{-1} = (-1)^{\binom{n}{2}} *. \tag{2.4}$$

2.2 The co-derivative

The interior derivative of a scalar-valued differential form u is defined in the literature as $*^{-1}d * u$ or as $* \ d * u$, which may differ by the sign.

We retrospectively choose the first of these options, for the same reason that Kähler must have chosen it. Indeed, let us compute. The formula for the exterior derivative of the Clifford product $u \vee z$ reduces to just two terms:

$$d(u \vee z) = du \vee z - \eta d_h \vee e^h z. \tag{2.5}$$

Hence, it is clear that

$$*^{-1} d * u = (-1)^{\binom{n}{2}} d(u \vee z) \vee z = du \; - \; (-1)^{\binom{n}{2}} \eta d_h u \vee e^h z \vee z. \tag{2.6}$$

On the other hand,

$$(-1)^{\binom{n}{2}} e^h z \vee z = (-1)^{\binom{n}{2}} \omega^h \vee z \vee z = \omega^h. \tag{2.7}$$

Hence,

$$*^{-1} d * u = du \; - \eta d_h u \vee \omega^h \tag{2.8}$$

Consider next the dot product $\omega^h \cdot d_h u$,

$$2\omega^h \cdot d_h u = \omega^h \vee d_h u - \eta d_h u \vee \omega^h. \tag{2.9}$$

We thus have

$$*^{-1} d * u = du + 2\delta u - \partial u = \delta u. \tag{2.10}$$

We have thus proved that the co-derivative of a scalar-valued differential form is nothing but the inverse Hodge dual of the exterior derivative of the Hodge dual. Notice that, in proving this result, we did not need to assume anything about the specific valued ness of u. The Hodge dual here is defined in the Kähler algebra.

2.3 Laplacians

The Laplacian of a differential $r-$form is defined as $\partial\partial$. We then have

$$\partial\partial = (d + \delta)^2 = dd + d\delta + \delta d + \delta\delta. \tag{2.11}$$

In general, none of these terms can be ignored. However, one is used to see simpler forms. Indeed, if u is scalar valued, $ddu = 0$. On the other hand, we have, with an abuse of parentheses for greater clarity,

$$\delta\delta u = \left(d\left\{[d(uz)]\, z^{-1}\right\} z\right) z = (dd(uz))\, z. \tag{2.12}$$

If $ddu = 0$, then $\delta\delta u$ also is zero. Hence the Laplacian of scalar-valued differential forms can be written simply as

$$\partial\partial = d\delta + \delta d. \tag{2.13}$$

Since the interior derivative of scalar-valued differential forms is zero, the Laplacian for these forms is then given by just the term δd. This is similar for what in the vector calculus is the divergence of the gradient.

Exercise. Show that $\partial\partial r = \frac{2}{r}$.

3 Harmonic, strict harmonic and constant differentials

We now deal with certain types of differential forms of special interest from a perspective of differentiation. We go from the east to the most specialized.

3.1 Harmonic and strict harmonic differentials

A differential form α is called harmonic if its Laplacian is zero. It is called *strict harmonic* if $\partial\alpha = 0$. The statement $\partial\alpha = 0$ is clearly equivalent to $d\alpha + \delta\alpha = 0$. Clearly, a strict harmonic differential is harmonic. In general, harmonic differentials are not strict harmonic. There are exceptions. If a differentiable manifold is oriented and compact and its metric is positive definite, harmonicity implies strict harmonicity (See Kähler 1962, section 21). Hence, in Euclidean spaces, finding harmonic and strict solutions is the same problem. Structurally, the problem of finding solutions of $\partial\alpha = 0$ is cleaner than finding solutions of $\partial\partial\alpha = 0$. But, the richness of solutions is such that one needs far more theory than this in order to solve the problem without resorting to the method of separation of variables.

Strict harmonic differential forms play a major role in the subalgebra of even differential forms of the Kähler algebra of 2-D Euclidean space.

As an immediate consequence of the definitions, an even differential form $u + vdxdy$ is strict harmonic if and on it satisfies the Cauchy-Riemann conditions

$$u_{,x} = v_{,y} \qquad\qquad u_{,y} = -v_{,x}\,. \tag{3.1}$$

In particular, $x + ydxdy$ is harmonic. It is an immediate consequence of these equations that the equations

$$U = \int udx - vdy, \qquad\qquad V = \int udy + vdx \tag{3.2}$$

define forms U and V, since the integrability conditions are the Cauchy-Riemann equations. We thus have

$$dU = udx - vdy, \qquad\qquad dV = udy + vdx \tag{3.3}$$

and, therefore,

$$U_{,x} = u = V_{,y} \qquad\qquad U_{,y} = -v = -V_{,x}\,. \tag{3.4}$$

These equations for the partial derivatives of U and V take the form of Cauchy-Riemann conditions. Hence, the differential forms $U + Vdxdy$

also are strict harmonic. U and V are determined only up to additive differential forms whose covariant derivative is zero, since they do not change the Kähler derivative.

Functions of strict harmonic differential forms need not be strict harmonic, but most common functions are. Whether they are or not is the same issue in the literature of whether or not a function of a complex variable is analytic or not.

3.2 Constant differentials

It seems that the term *constant differential form* has not been used by anybody but Kähler. So his use is justified when it does not overlap and conflict with prior use of the same term in the literature. His choice of that term is not very fortunate in that he uses it not only for scalar-valued differential forms —where there is no problem— but also for tensor-valued differential forms, where the conflict arises.

Define constant differential forms, c, as those whose covariant derivatives $d_h u$ are zero. This is a more restrictive concept than $\partial u = 0$. It is easy to show that any exterior polynomial with constant coefficients in the differentials of the Cartesian coordinates is a constant differential form, but not if those coefficients are not constant. This is easy to prove by building the argument upon observation of what happens when we build the differentials of, say, $f(x,y)dx \wedge dy$ and $f(x,y,z)dx \wedge dy$. As for polynomials on the differential of other coordinates, suffice to observe that dr has interior derivative $2/r$. Hence $d_h r$ cannot be zero. But this does not impede that polynomials in those differentials with non-constant coefficients may happen to be constant differentials. An example of this is $\rho d\rho d\phi$, which is the unit surface element, $dxdy$, but expressed in terms of cylindrical coordinates. The *unit volume differential* is a constant differential, since we can always choose coordinates such that at any specific point they behave as if they were Cartesian coordinates, this is to say that the metric reduces to the form $\sum(dx^i)^2$ at the given point.

The rules for the different versions of the Leibniz differentiation readily yield

$$d(u \wedge c) = du \wedge c, \qquad \partial(u \vee c) = \partial u \vee c, \tag{3.5}$$

$$\partial(u \wedge c) = \partial u \wedge c + \eta d_h u \wedge e^h c, \qquad d(u \vee c) = du \vee c - \eta d_h u \vee e^h c. \tag{3.6}$$

The equation $\partial(u \vee c) = \partial u \vee c$ is a most important one vis a vis the foundations of quantum mechanics. Recall that we had the equation

$$u = {}^{+}u^{+}\, \epsilon^{+}\tau^{+} + {}^{+}u^{-}\, \epsilon^{+}\tau^{-} + {}^{-}u^{+}\, \epsilon^{-}\tau^{+} + {}^{-}u^{+}\, \epsilon^{-}\tau^{-}, \tag{3.7}$$

whose importance cannot be overemphasized The operators $\epsilon^{\pm}$, $\tau^{\pm}$ and $\epsilon^{\pm}\tau^{*}$ are constant idempotents. Hence, we have in particular

$$\partial u = \ (\partial\ ^{+}u^{+})\ \epsilon^{+}\tau^{+} + (\partial\ ^{+}u^{-})\ \epsilon^{+}\tau^{-} +\ (\partial^{-}u^{+})\ \epsilon^{-}\tau^{+} +\ (\partial^{-}u^{+})\ \epsilon^{-}\tau^{-}, \tag{3.8}$$

and similarly for decompositions involving only the $\epsilon^{\pm}$ or the $\tau^{\pm}$. Let us post-multiply by $\epsilon^{+}\tau^{+}$. We get

$$(\partial u)\epsilon^{+}\tau^{+} = \ (\partial\ ^{+}u^{+})\epsilon^{+}\tau^{+}. \tag{3.9}$$

We could similarly have multiplied by any other of the $\epsilon^{\pm}\tau^{*}$ and have obtained a parallel result. Because of the form of the Kähler calculus, profound implications of this are around the corner. Several applications will be shown in the next chapter.

Consider now the metric tensor, $g_{ij}\mathbf{e}^{i}\otimes\mathbf{e}^{j}$, in order to illustrate the issue of terminology to which we referred above. Under Kähler's terminology, this tensor field is a constant tensor-valued differential $0-$form. He could also consistently call constant tensor field since he uses tensor-valued differential $0-$form and tensor field as synonymous. So, we would be using the term constant tensor field for tensor fields which are not constant. We have described the actual situation in a simplified way since, for his argument, he actually creates a tensor field that he denotes as $(dx \vee dx)$ whose components are given by $(dx \vee dx)^{ik} = dx^{i} \vee dx^{k}$. Be as it may, the fact remains that he uses the term constant to refer to something where nothing appears to be constant; the constancy is a combination of terms, derivatives among them. Fortunately, we do not have to deal with tensor-valuedness in this course. So, we do not have to worry with matters of notation.

4 Application 1: Helmholtz theory in 3-D Euclidean space

We now derive the differential $1-$forms version of Helmholtz theorem. It amounts to the integration of a differential $1-$form whose exterior and interior derivatives are given and satisfies the right conditions. We then proceed to obtain from this result the integration of a differential $2-$form also given by its exterior and interior derivatives. Similar theorems in non-Euclidean spaces and higher dimensions will be proved in later chapters, where we prove a uniqueness theorem which is required as a prerequisite for the validity of the proof. this requirement also applies here. but we proceed without it since the results parallel those of Helmholtz theorem in the vector calculus where the uniqueness theorem is well known. So, for the time being, we trust that such a theorem also exists for differential $1-$forms. The contents of this section, is virtually the same as in the substantive part of our paper "Helmholtz theorem for differential forms in 3-D Euclidean space", posted in arXiv. We would not reproduce it here except for the following.

Readers of books dealing with differential forms will have noticed that they often have a $\frac{1}{r!}$ at the front of expressions. This is because those are summations over all permutations of indices. As an example, we cannot write $dx^1 \wedge dx^2$ as $\sum dx^i \wedge dx^j$ with $i = 1, 2$ and also $j = 1, 2$. We have to introduce the factor 1/2 and the minus sign, i.e. $dx^1 \wedge dx^2 = (1/2)(x^1 \wedge dx^2 - x^1 \wedge dx^2)$. Once we have taken care of this issue, we still have to contend with the fact that we may have summation over the different elements in a basis of $r-$forms. This is taken care of with the convention that summations over repeated indices are only over independent elements, meaning that the notation applies to the form that differential forms take after we have reduced say $(1/2)(x^1 \wedge dx^2 - x^1 \wedge dx^2)$ to $dx^1 \wedge dx^2$. Once we have done it, we still have other $dx^i \wedge dx^j$ pertaining to other pairs of indices to cover all options when dealing with differential $r-$forms. And then we may have to sum over the different grades. So, we shall use the notation $a_R dx^R$ to sum from $R = 1$ to $R = 2^n$, which is the dimension of the algebra. If some u is a differential $r-$form, it means that all a_R are zero except, at most, a number $\binom{n}{r}$ of them, which is the dimension of the subspace of differential $r-$forms. In this section, we shall acquire plenty of practice with this.

Finally, we have virtually shown earlier in this chapter that the definition of the Laplacian of a scalar-valued differential $1-$form yields an operator that is the same as in the standard vector calculus. We thus take advantage of knowledge by readers of, for example, what the Laplacian of $\partial\partial\frac{1}{r}$ is.

4.1 Helmholtz Theorem for Differential 1-Forms in 3-D Euclidean Space

We compute with Cartesian coordinates. The results obtained being invariant, they retain their form in arbitrary bases of differential forms.

Let w denote the unit volume differential form in dimension three. (We reserve the letter z for arbitrary dimension). And let us define r_{12} as

$$r_{12} = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-1/2}$$

Theorem: Differential $1-$forms that are smooth and vanish sufficiently fast at infinity can be written as

$$\alpha(\boldsymbol{r}) = -\frac{1}{4\pi} dI^0 - \frac{1}{4\pi}\delta(dx^j dx^k I^i), \tag{4.1}$$

$$I^0 \equiv \int \frac{1}{r_{12}}(\delta'\alpha')w', \qquad I^i \equiv \int \frac{1}{r_{12}} d'\alpha' \wedge dx'^i, \tag{4.2}$$

with summation over the three cyclic permutations of 1,2,3.

Proof: By the uniqueness theorem and the annulment of dd and $\delta\delta$, the proof reduces to showing that δ and d of espectively I^0 and I^i yield $d\alpha$ and $\delta\alpha$.

Since $\delta dI^0 = \partial\partial I^0$, we write $-(1/4\pi)\delta dI^0$ as

$$\frac{-1}{4\pi}\partial\partial I^0 = \frac{-1}{4\pi}\int_{E'_3}(\partial\partial\frac{1}{r_{12}})(\delta'\alpha')w' = \frac{-1}{4\pi}\int_{E'_3}(\partial'\partial'\frac{1}{r_{12}})(\delta'\alpha')w' = \delta\alpha, \tag{4.3}$$

after using the relation of $\partial\partial$ to the Dirac distribution.

For the second term, we use that $d\delta = \partial\partial - \delta d$ when acting on $dx^j dx^k I^i$. We move $\partial\partial$ past $dx^j dx^k$. Let α be given as $a_l(x)dx^l$ in terms of the same coordinate system. We get $d'\alpha' \wedge dx'^i = (a'_k,_j - a'_j,_k)w'$. The same property of $\partial\partial$ now allows us to obtain $d\alpha$.

For the second part of the second term, we apply δd to $dx^j dx^k I^i$:

$$\delta d(dx^j dx^k I^i) = \delta\left(w\frac{\partial I^i}{\partial x^i}\right) = wd\left(\frac{\partial I^i}{\partial x^i}\right) = wdx^l\frac{\partial^2 I^i}{\partial x^i \partial x^l} =$$

$$= wdx^l \int_{E'_3}\left[\frac{\partial^2}{\partial x'^i \partial x'^l}\left(\frac{1}{r_{12}}\right)\right](a'_k,_j - a'_j,_k)w'. \tag{4.4}$$

We integrate by parts with respect to x'^i. One of the two resulting terms is:

$$wdx^l \int_{E'_3}\frac{\partial}{\partial x'^i}\left[\frac{\partial\left(\frac{1}{r_{12}}\right)}{\partial x'^l}(a'_k,_j - a'_j,_k)\right]w'. \tag{4.5}$$

Application to this of Stokes theorem yields

$$wdx^l \int_{\partial E'_3}\frac{\partial\left(\frac{1}{r_{12}}\right)}{\partial x'^l}(a'_k,_j - a'_j,_k)dx'^j dx'^k. \tag{4.6}$$

It vanishes for sufficiently fast decay at infinity.

The other term resulting from the integration by parts is

$$-wdx^l \int_{E'_3}\frac{\partial\left(\frac{1}{r_{12}}\right)}{\partial x'^l}\frac{\partial}{\partial x'^i}(a'_k,_j - a'_j,_k)w', \tag{4.7}$$

which vanishes identically (perform the $\frac{\partial}{\partial x'^i}$ differentiation and sum over cyclic permutations). The theorem has been proved.

4.2 Helmholtz Theorem for Differential $2-$forms in E_3

The theorem obtained for differential $1-$forms, here denoted as α, can be adapted to differential $2-$forms, β, by defining α for given β as

$$\alpha \equiv w\beta, \qquad \beta = -w\alpha. \tag{4.8}$$

Then, clearly,

$$w\delta(w\beta) = -d\beta, \qquad wd\beta = \delta(w\beta). \tag{4.9}$$

Helmholtz theorem for differential $1-$forms can then be written as

$$w\beta = -\frac{1}{4\pi}d\left(\int_{E_3}\frac{\delta'(w'\beta')}{r_{12}}w'\right) - \frac{1}{4\pi}\delta\left(dx^{jk}\int_{E_3}\frac{d'(w'\beta')\wedge dx'^{i}}{r_{12}}\right), \tag{4.10}$$

and, therefore,

$$\beta = \frac{1}{4\pi}wd\left(\int_{E_3}\frac{\delta'(w'\beta')}{r_{12}}w'\right) + \frac{1}{4\pi}w\delta\left(dx^{jk}\int_{E_3}\frac{d'(w'\beta')\wedge dx'^{i}}{r_{12}}\right). \tag{4.11}$$

The integrals are scalar functions of coordinates x. We shall use the symbol f to refer to them in any specific calculation. In this way, steps taken are more easily identified.

The first term in the decomposition of β, we transform as follows:

$$wdf = (\partial f)w = \partial(fw) = \delta(fw), \tag{4.12}$$

where we have used that w is a constant differential.

For the second term, we have:

$$w\delta(dx^{jk}f) = w\partial[wdx^{i}f)] - wd[fdx^{jk})]. \tag{4.13}$$

The first term on the right is further transformed as

$$w\partial(wdx^{i}f) = wwdx^{i}\partial f = -dx^{i}df, \tag{4.14}$$

where we have used that wdx^{i} is a constant differential, which can be taken out of the ∂ differentiation. For the other term, we have

$$-wd(fdx^{jk})] = -wdf\wedge dx^{jk} = -wf_{,i}\,w = f_{,i} = dx^{i}\cdot df. \tag{4.15}$$

From the last three equations, we get

$$w\delta(dx^{jk}f) = -dx^{i}df + dx^{i}\cdot df = -dx^{i}\wedge df = d(dx^{i}f). \tag{4.16}$$

In order to complete the computation, we have to show that $d(w\beta) \wedge dx^i$ can be written as $\delta\beta \wedge dx^{jk}$. This can be shown easily by direct calculation. Let α be given as $a_i dx^i$. Then $d(w\beta) \wedge dx^1 = d\alpha \wedge dx^1 = (a_{3,2} - a_{2,3})w$. On the other hand, $\beta = -a_i dx^{jk}$ and

$$\delta\beta = (a_{3,2} - a_{2,3})dx^1 + \textit{cyclic permutations}. \tag{4.17}$$

Hence $\delta\beta \wedge dx^{23} = (a_{3,2} - a_{2,3})w$ and, therefore,

$$d(w\beta) \wedge dx^1 = d\alpha \wedge dx^1 = \delta\beta \wedge dx^{23}, \tag{4.18}$$

and similarly for the cyclic permutations of the indices.

5 Application 2: Cauchy like calculus with real differential forms

In chapter 2, we dealt with algebraic issues which were sufficient by themselves to compute without resort to complex variable theory certain types of real integrals which are usually solved using that theory. We shall now produce theory that involve operations which differential forms and that give results as if we were computing complex integrals.

When this author did all this work on this subject (two papers in arXiv), it was not too clear to him what was the most important feature of what he was doing. He was distracted by whether an analytic function was defined by a power series (Weierstrass point of view) or by compliance with the Cauchy-Riemann conditions (Cauchy's point of view). But the difference is more profound; it lies somewhere else. We proceed to explain this at length in subsection 5.3, after we had introduced the basis concepts.

The theory of complex variable not only deals with the solution of those real integrals, but is also used to solve integrals which are not real. The question then is whether one can extend the theory of real differential forms so that it also comprises this new situation. One can extend it, and that is what this section is about.

5.1 Representation with real differential forms of complex integrals

This section supersedes subsections 2.3 and 3.2 of our paper "Real Version of Calculus of Complex Variable: Cauchy's Point of View". That subsection 2.3 is confusing and may be even misleading. It will now be made unnecessary. As for subsection 3.2, it is clear but irrelevant since it does not clearly address the issue that will be considered here under the term antivaluation. This is the operation opposite of valuation, rightly considered there as in here. We think that its naturalness is here made much more clearly.

5.1.1 Valuations

The equivalence between i and $dxdy$ is only algebraic. The unit i is a constant and, as such, it can be taken out of an integral sign. But one cannot do so with $dxdy$, since it would change the nature of the integral. So, one has to introduce a theory with real differential forms that represents the calculus of complex variable through expressions where i is outside the integral. This takes place as follows.

With $f(z) = u + iv$ and $dz = dx + idy$ (which is correct, but which we might also write as $\partial z = \partial x + i\partial y$), we have

$$\int f(z)dz = \int f(z)dx + i\int f(z)dy = \int (u+iv)dx + i\int (u+iv)dy =$$
$$= \int udx - vdy + i\int udy + vdx = U + iV. \tag{5.1}$$

for analytic functions, i.e. satisfying the Cauchy-Riemann conditions. Hence, we ignore the first equalities and focus on the last one. In this section, we use the symbol w to refer to $u + vdxdy$. The last equation can then be written as

$$\int udx - vdy + dxdy\int udy + vdx = U + Vdxdy, \tag{5.2}$$

We replace integration $\int_c f(z)dz$ on a curve of the complex plane with "valuation" $\langle w\rangle_c$ of an edif, $u + vdxdy$, on a curve c of the real plane:

$$\langle w\rangle_c \equiv \left[\int_c wdx\right] + dxdy\left[\int_c wdy\right], \tag{5.3}$$

The integrability conditions for these integrals to not depend on c but only on its end points are the Cauchy-Riemann relations. For $u+vdxdy$, this is the strict harmonic differential condition. "Valuation potentials"

$$\langle w\rangle = U + Vdxdy = \int udx - vdy\ + dxdy\int (udy + vdx) \tag{5.4}$$

then exist. Of course, U and V are undefined up to integration constants. The valuation on a closed curve (on simply connected manifolds) is zero. This is the Cauchy-Goursat theorem for strict harmonic differentials. In domains that are not simply connected, we surround the poles enclosed by closed curves C with equally oriented circles c_i, all of them with the same orientation as C and containing one and only one pole each. We then have

$$\langle w \rangle_C = \sum_i \langle w \rangle_{c_i} . \tag{5.5}$$

From now on, let us denote $U + V dxdy$ as W. We have

$$\partial W = dU + dV dxdy. \tag{5.6}$$

Since

$$dU = udx - vdy, \qquad dV = udy + vdx, \tag{5.7}$$

we obtain

$$U_{,x} = u = V_{,y} \qquad U_{,y} = -v = V_{,x} \ . \tag{5.8}$$

W is, therefore, strict harmonic. Notice that we have not spoken of $f(x+ydxdy)$ inside of an integral sign. The Cauchy theory of differential forms belongs to the elements of the algebra of even differential forms, not to the functions of some other variable.

5.1.2 Antivaluations

The valuation plays the role played by integration in the calculus of complex variable. We shall refer to $u + vdxdy$ as the antivaluation of $U + V dxdy$. It is a simple matter to show that

$$dxdU = dydV \qquad dydU = -dxdV, \tag{5.9}$$

and

$$dUdx = dVdy \qquad dUdy = -dVdx. \tag{5.10}$$

The antivaluation can be given in a variety of ways:

$$u = dx \cdot dU = dy \cdot dV = idy \wedge dU \ = -idx \wedge dV \tag{5.11}$$

and

$$v = dx \cdot dV = -dy \cdot dU = idx \wedge dU \ = idy \wedge dV. \tag{5.12}$$

i referring, of course, to $dxdy$.

To strengthen ideas, let us see what $f(z) = z = x + iy$, $df/dz = 1$ corresponds to. We compute the antivaluation for $U = x$, $V = y$. We get $u = dx \cdot dU = dx \cdot dx = 1$ and $v = dx \cdot dV = dx \cdot dy = 0$. So, $x+ydxdy$ is a primitive of 1. We check it computing the valuation:

$$\int 1dx + dxdy \int 1dy = x + ydxdy. \tag{5.13}$$

5.1.3 Cauchy calculus versus exterior and Kähler calculi

The contents of the previous two subsubsections can be taken as the foundations of a Cauchy calculus of strict harmonic differential forms. The Cauchy calculus is different from the Kähler calculus of differential forms, since it is based on a different concept of integration and differentiation. What goes by the concept of integration and differentiation in the calculus of complex variable has a correspondence in valuations and antivaluations in the Cauchy calculus of differential forms. The fact that this calculus uses the Kähler algebra might lead one into confusion. Readers might be right in thinking that this section belongs to the previous chapter. This is logically correct, but consider this. A Kähler calculus on a Kähler algebra is far more important than a Cauchy calculus on the same algebra. It is better to be confused learning this Cauchy calculus after having learned the Kähler calculus than to be confused learning the latter after having learned the former.

Let me insist on the point just made. In the Kähler calculus, differential $r-$forms are evaluated (i.e. integrated) on $r-$surfaces. In the Cauchy calculus of differential forms presented so far, we "valuate" even differential forms on curves. The valuation on curves involves the integrating or evaluation on those curves of associated differential $1-$forms, specifically the differential $1-$forms $udx - vdy$ and $udy + vdx$.

$$dU = udx - vdy, \qquad\qquad dV = udy + vdx. \tag{5.14}$$

Properly speaking, an expression such as $dz = dx + idy$ does not pertain to the complex variable calculus proper, but to the representation in the plane of differential forms over the complex numbers. The obtaining of dz through differentiation belongs to the exterior calculus of these differential forms, not to the calculus of complex variable, where differentiations take place with respect to z.

Let us use define $Z = x+idxdy$. It is certainly the case that $dz = dx+idy$ and that, in contrast, $dZ = dx$. But, as we have explained, $d(z) = dx + idy$ is not a differentiation in the calculus of complex variable and $d(Z) = dx$ is not a differentiation in the Cauchy calculus of differential forms. We cannot make anything of this difference.

The differentiation of z in the calculus of complex variable is simply dz. End of story. We can of course write $dz = (dz/dz)dz = dz$. In the calculus of complex variable, the differentiation of x takes place through the differentiation of $x = (z+z^*)/2$. This poses problems when trying to differentiate z^*, for which purpose one resorts to a representation in the plane. See, for instance, H. Cartan's "Elementary Theory of Analytic Functions of One or Several Complex Variables". He introduces z and z^* for the first time in his book as follows: "Recall that a complex number

$z = x + iy$ ... is represented by a point on the plane ... If we associate with each complex number $z = x + iy$ its 'conjugate' $\bar{z} = x - iy$...". Of course, H. Cartan knows better. Later in his book he considers differentiations with respect to the complex variables z and its conjugate $\bar{z}$ and introduces the Laplacian as $\partial^2 f/\partial z \partial \bar{z}$. And later, when dealing with n complex variables, he writes

$$df = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial z_k} dz_k + \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k \right), \tag{5.15}$$

for the differential of a function of so called n complex variables. When this is done, $d(x+iy) = dz$ through a process where x and y are expressed in complex variable terms.

This discussion was worth writing for an indirect purpose: as reformulated, the exterior calculus of one complex variable can be seen as a theory of strict harmonic differentials. Forget about functions of $x + iy$. That may be included in the game, specially as a bridge to start the game, like even the most notable mathematicians do. But, once again, we did not use $x+ydxdy$ above. And, if one uses it and the likes of it, we shall have to take into account that the relation between "the complex variables" $x + ydxdy$ and $x + zdxdz$ will not be the same as the relation between $x + ydxdy$ and $z + tdzdt$, assuming of course positive definite metric. Indeed $dxdydxdz$ equals a differential $2-$form; $dxdydzdt$ is a differential $4-$form. Its interior derivative is not a differential $1-$form.

We have set the stage for a Cauchy theory involving strict harmonic differential forms in Euclidean spaces of arbitrary dimension, not spaces of several complex variables. It is a theory much richer than standard theory of several complex variables.

5.2 Cauchy's theorems

This subsection should be about Cauchy's integral formula and Cauchy's integral formula for derivatives. They are clearly explained in subsections 3.1 and 3.3 of our paper "Real Version of Calculus of Complex Variable: Cauchy's Point of View". Although we have said that the Cauchy theory of differential forms belongs to the elements of the algebra of even differential forms, not to the functions of some other variable, using the notation $f(z)$ instead of $f(x + ydxdy)$, which in turn we use there instead of $u + vdxdy$, has notational advantages. The nature of those theorems is at the root of those notational advantages.

See the same reference for a couple of exercises.