RELATION OF PROFESSOR R. POLI'S KEYNOTE LECTURE TO THE KÄHLER CALCULUS

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The 1999 Lemelson-MIT Prize for invention and innovation was granted to Caltech's Emeritus Professor Carver Mead. Upon receiving this prize, he pronounced a five paragraph acceptance speech full of wisdom. We cannot repeat it here for lack of space. We can, however, extract a number of sentences that give the gist of what he said about present day knowledge and the evolution of the culture:

> "How do we, as a human culture, prepare, ... for this world in which the knowledge base turns over many times within a simple human lifetime?
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> One answer to this dilemma is specialization: One can become an expert... learn more and more about less and less, until eventually, we know everything about nothing! ... If specialization were to be our only response to rapidly evolving knowledge, I would view our prospects as a culture with deep concern, even with alarm.
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> ...Arthur Koestler defines the creative process as starting with the juxtaposition of two concepts from two separate conceptual spaces. If we, ... , restrict the development of individual talents to narrow specializations, we will thereby lose the ability to innovate.
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> ... there is, within our culture, an evolution of knowledge over and above the addition of facts and the specialized understanding of those facts. Many phenomena that in the past were seen as separate are now understood to be the same... Each of these equivalences represent a major *unification* and *simplification* of the knowledge base. Ideas formerly occupying separate conceptual spaces now occupy the same conceptual space.
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> It is this unification and simplification of knowledge that gives us hope for the future."[1].

Erich Kähler's march towards impredicative quantum physics

In a 1960 paper [2], Kähler produced a generalization of the exterior calculus and proposed the equation

$$\partial u = au \tag{1}$$

as alternative to the Dirac equation. ∂ is a differential operator. At its most general, his calculus is about what he called differential tensors, or tensor-valued differential forms in modern mathematical language. But Eq. (1) was meant only for scalar-valued differential forms a and u. Kähler did not do anything with this equation. He referred to it as the Dirac equation, even though it is far more general because one can make the input a whatever we want and also because u is not confined to be a member of an ideal (read spinor). Notice its formal parallelism of (1) with the fruitful equation $y' = ay$ of the ordinary calculus..

In a 1961 paper [3], he defined idempotents for time translation and rotational symmetries. These idempotents are of a type he calls constant differential forms. They have the property that a solution of (1) in the algebra can be decomposed into a sum of solutions in the left ideals that those idempotents define. If these "spinor solutions" satisfy

Date: March 16, 2017.

the appropriate additional properties, they represent positrons and electrons. With such spinors and with $a = m + eV(r)dt$, where $V(r)$ is the Coulomb potential, he solved the fine structure of the hydrogen atom (we are ignoring constants $\hbar$, c and i, which are irrelevant for the present discussion).

His 1962 paper, [4], is a more comprehensive version of the one from 1960. Among a variety of new topics, there is his derivation of the conservation law, using the first Green identity. The amplitude for charge emerges, with its two signs. He again solved the hydrogen atom, now more efficiently than in 1961. He also proposed an awful Kähler equation for non-scalar valued input. Without going into details, let us simply give it here

$$(\partial u)_{i\ldots j}{}^{k\ldots l} = a_{i\ldots j}{}^{k\ldots lp\ldots q}{}_{r\ldots s}\, u_{p\ldots q}{}^{r\ldots s}. \tag{2}$$

The two series of indices in ∂u have to do with contravariant and covariant tensor valuedness. We are not advocating the use of this equation, for which Kähler himself did not give any application. But it speaks of the enormous potential that going beyond scalar-valuedness may provide. With such valuedness anchored in spacetime, one can go into physical theory far beyond what one can do with spacetime based Dirac equations.

In a 1992 paper, [5], Kähler returned to this calculus with a paper titled "Raum-Zeit Individuum" (Space-time Individual"). He proposed equation (1), but with a constant ρ replacing a as input: $\partial u = \rho u$. He made the point that one should use it as a "universal" equation (we prefer to refer to it as "mother equation" in view of developments to follow.

It seems that the use of the term "Individuum" seeks to emphasize the specialization necessary for applications of his universal equation, which he did not provide in any case. He failed to show any applications of the computations he developed, but his proposal has merit and can be implemented. It is worth exploring this because, correct or not in its details, it opens new avenues for research. Any success, however limited, resulting from such avenues would represent a contribution to the unification of conceptual spaces.

Ab initio, an undertaking of the nature of what Kähler tried to do in 1992 may look very ambitious, even too ambitious. But this author has found it to be feasible, with an equation very similar to (1) but more flexible and promising. In our presently incomplete "Kähler's quantum physics" paper (posted in this web site), we shall speak of a mother equation $\partial u = u$ that will play the role that Kähler intended for his $\partial u = \rho u$ equation. In its section 4, we shall show structural relations between such an equation and those of the London brothers, on the one hand, and the Landau-Ginzburg equations, on the other hand.

Of impredicativity and external inputs

The term impredicativity is not found in ordinary dictionaries, where the term predicative is found simply as an adjective associated with predicate. In logic, predicate is something that can be affirmed or denied about a subject of a proposition. We would consider the equation $\partial u = u$ to be impredicative, since it is a self-referential proposition. Indeed, we not only find u on the left hand side, where one usually finds outputs, but also on the right hand side, where, in addition, it stands alone. Compare this with Maxwell's equations, which can be written as $\partial F = j$. So, the issue is whether there is an interplay between the concepts of impredicativity and external input. How should the term impredicative be understood when dealing with physics equations? This is an issue that we physicists and mathematicians should resolve, but we can certainly get some guidance from philosophers of science like Professor Poli.

The "very impredicative" equations that we are going to be dealing with are formidable and, given the state of the art, are will remain so in a foreseeable future. Indirect support for daring equations comes from the side of daring types of solutions to accompany them.

Kähler went as far as we think it was possible to go in physics with scalar-valuedness. Solutions of more comprehensive Kähler equations —for example if they were Clifford-valued— would take a form that we have explored in papers posted in arXiv in recent years. We found that there is an algebraic representation of quarks which may become very significant if further developed with help of available phenomenology. External inputs are at least required due to the actual comprehensiveness of such simple looking equations.

Post-scalar valuedness introduces geometric considerations into this discussion, since ∂ may not only depend on the Christoffel symbols, but also on the "affine connection", which, in general, will not have those symbols as components. Hence, ∂ also is an external input, unless one considers systems of equations whose output is not only u but also the differential invariants that define the spacetime manifold. External input must nevertheless be present somewhere, or else we would learn less and less about more and more, which is as undesirable as learning more and more about less and less. We shall now see that, in principle, such systems with "minimal external input" may exist. The systems that we are about to discuss represent reality, but that is a second order issue for present purposes.

On an Einstein theory, jointly with Kähler's

In the late 1920s, Einstein postulated teleparallelism (here synonymous with zero affine curvature, $\mho = 0$) for unification. He never said explicitly what he expected to unify. But he certainly was thinking of unifying gravitation and electrodynamics, since he sought to supersede his 1915 theory, and he expressed disappointment at not having found Maxwell's equations. He did not know the required mathematics, not even to develop $\mho = 0$. Part of the requirement is the use of the not yet discovered Finslerian bundles and connections on them. But that could not cover quantum mechanics at all.

For the moment, consider the system

$$\partial u = au, \qquad \mho = 0, \tag{3}$$

where $\mho$ is of the type called, in the literature, metric compatible affine connection. A more appropriate term is Euclidean or pseudo-Euclidean connection. Simultaneously with null Euclidean curvature, there will still be Riemannian curvature, in general different from zero and directly related to metric relations, but not to comparison of vectors at a distance. If we make a equal to the unity in this system, only u is not directly related to the structure of the differentiable manifold, while ∂ and $\mho$ are. Once again, whether this system actually represents (granted, incompletely) the basic fields of physics is important, but not as much as the fact that, when developed, it parallels main lines of the present paradigm. Also, we shall later introduce a more complete system of equations.

In section 5 of our aforementioned paper, we shall show how easily the first pair of Maxwell's equations emerges from the first Bianchi identity (Of course, this assumes that we identify the components of the electromagnetic field with certain components of the torsion). For this emergence to make sense, we have to assume that the base space for the connection is not space-time but what Cartan [6], in dealing with classical mechanics, called the space of elements. It is generated by the differentials dt, dx^i and dv^i. The $v^{i\prime}$s are coordinate functions. The evaluation of the dv^i's on so called natural liftings of spacetime curves become the velocity components of any particles whose motion would be represented by such curves. The space of elements actually is the base space for Finslerian connections, which may exist in Riemannian and pseudo-Riemannian metrics.

In teleparallelism, there is also a geometric version of electromagnetic energy-momentum current, which is a vector-valued differential 3-form improperly called tensor. I had not noticed until recent years that there also is a locally meaningful gravitational "energy-momentum tensor", as legitimate as the electromagnetic one. There is not even the opportunity, much less the need, for the concepts of energy-momentum pseudo tensors and non-locality of gravitational energy-momentum to arise. The energy-momentum current is missing in the original Einstein theory (1915), and is incompatible with the extension of Riemannian geometry that took place in 1917 with the emergence and addition to Riemannian geometry of the concept of connection, specifically the Levi-Civita (LC) connection.

We are fully aware of the fact that the LC connection has the Christoffel symbols as components, which were half a century old when that connection was born. But the role of those symbols had nothing to do with connections, Levi-Civita's being the first one to be considered by mathematicians. No comparison of tangent vectors at different points of the spacetime manifold, whether path-dependent or not, existed until then. Connections are precisely what determines how we compare vectors at a distance. The LC connection was independently and almost simultaneously discovered by Levi-Civita, Hessenberg and Schouten, which seems to indicate that its time had arrived. Of course, there is an infinite number of connections compatible with a given Riemannian or pseudo-Riemannian metric, but this was not known until Cartan showed it in the early 1920s.

Within a few years after 1917, general relativity adopted the so enriched Riemannian geometry. The LC connection thus entered general relativity. The equations of the geodesics (meaning extremals) became at the same time the equations of the autoparallels (meaning lines of constant direction). But the LC was not the right connection for the geometrization of electrodynamics. The autoparallels in Finsler geometry give rise to the equations of motion of particles in general relativity together with additional terms that amount to the Lorentz force. But the geometrization of electrodynamics through teleparallelism and the Finsler bundle is not enough. The second pair of Maxwell's equations does not appear anywhere in the structure just described. They should be implied by quantum mechanical equations of any system in question.

Anticipation and impredicativity in a classical/quantum system of physical equations.

A very important characteristic of the system of equations 2 is its being self referential with respect to the differential invariants that define the differentiable manifold, as both ∂ and $\mho$ are defined in terms of those invariants, and which one seeks to find out as solutions of differential systems. Thus both ∂ and $\mho$ constitute simultaneous input and output in the system of self- referential equations in which we shall be interested, which momentarily is (2). And so does u, though in a different way, as will be seen further down. Let us advance once again that, at some point, there should be some input in our differential system in order to specify which material system we may be interested in each particular case. In other words, we need something that will do what a in equation 1 does.

The concept of ∂ presupposes a metric. And we want $\mho$ to be the torsion of a Euclidean connection. Metric compatibility forces a restriction of the bundle of frames, from an affine bundle to a Euclidean or pseudo-Euclidean bundle. We should thus speak of a Euclidean (or pseudo-Euclidean) connection, not of an affine connection. But there is no such restriction in the system (2), which thus is not self-contained.

A metric tensor is a symmetric, quadratic differential form. Such a tensor is not a Clifford-valued differential form. But there is room for an equivalent null metric in the Kaluza-Klein space canonically determined by the spacetime manifold. The role of the metric is then played by the equation

$$d\wp(.,.)d\wp = 0, \tag{4}$$

where $d\wp$ is $\omega^{\mu}\mathbf{e}_{\mu} + d\tau\mathbf{v}$, where $\omega^{\mu}\mathbf{e}_{\mu}$ is the spacetime element; τ is a coordinate which, on spacetime curves, becomes propertime; and $\mathbf{v}$ is a vector in five dimensions whose projection upon spacetime coincides with the 4-vector velocity of particles whose motion is being described by those curves. It is clear that, through the components of that projection, this Kaluza-Klein space is intimately connected with the space of elements of classical mechanics, i.e. with the base space of the Finslerian geometry associated with spacetime. In view of what has been said, we replace the system (2) with the system

$$\partial u = u, \qquad \mho = 0, \qquad \partial\mathbf{\Omega} = 0 \qquad d\wp(.,.)d\wp = 0. \tag{5}$$

Here $\mathbf{\Omega}$ is the torsion.

Self- referential is a virtual synonym for impredicative, a concept that evokes by proxy other concepts like anticipation, dynamical systems, complexity, emergence, attractors, chaos, etc. Professor Poli is an expert and leader on the subject of impredicativity, which is why we invited him to deliver a keynote lecture on the subject. His talk meets our interests not only because of the subject of impredicativity, but also because he will also speak to us about another expertise of his, namely anticipation. We shall now see that the very flexible Kähler calculus allows us to implement anticipations, and attractors.

In the specialized literature, the term anticipation not only means what the dictionary defines as such, but there must also be a course of action to get to what is anticipated. Research is anticipatory if guided towards achieving the anticipated goal. Readers may retort that this is what physics researchers do all the time. True, but the lack of awareness of being doing it rather than seeking it actively makes their effort qualify as minor league anticipation. There are societal factors that impede most researchers to think big and act consequentially, not to mention the lack of appropriate mathematical tools to achieve the goal. If we want real progress in theoretical physics, thinking big should be promoted. But thinking big is necessary but not sufficient. There is no option but to try.

As an example, one could go a step further and think beyond just the unification of the interactions. One anticipates the relating of the mother equation not only to practical quantum physics but also to classical mechanics and the not-as-classical statistical physics. In the following, we shall try to make the case that this course of action is not implausible.

Of idempotents as attractors for physical differential systems

One is thinking big when one is able to mathematically develop an idea that looks impossible. If you are on the wrong track, mathematics will eventually catch up with you. Mathematics is the guardian of the consistency of daring ideas. Anticipatory work based on powerful mathematics and not just simplistic algebra should thus be had in high esteem, since it is subjected to powerful consistency tests. This does not impede coexistence of anticipation with healthy skepticism. I proceed to propose a couple of anticipatory ideas in mathematical physics that have never been explored, or which have never got the degree of diffusion that they deserve. They have to do with the work of Kähler in the first pages of [3].

The idempotents

$$\epsilon^{\pm} = \frac{1}{2}(1 \mp idt), ...I^{\pm} = \frac{1}{2}(1 \pm idxdy) \tag{6}$$

are directly and respectively related to charge and chirality, and almost as directly related to energy and spin, again respectively (The $I^{\pm}$ were designated as $\tau^{\pm}$ in Kähler's works and in previous publications by the present author. We cease to do so because we reserve the symbol τ for proper time).

In Kähler's theory, the products $\epsilon^- I^\pm$ are factors in the expressions for electrons with both chiralities, namely

$$e^{im\phi}e^{-iEt/\hbar}\; f(\rho, z, d\rho, dz)\epsilon^\pm I^*. \tag{7}$$

(The asterisk in I^*will mean a plus or a minus uncorrelated with the superscript in ϵ). Pairs of electrons with opposite chiralities could become paired and their sum would belong to the ideal defined by ϵ^-, since $I^+ + I^- = 1$. A Cooper pair would then be given by an expression of the form

$$e^{im\phi}e^{-iEt/\hbar}\; g(\rho, z, d\rho, dz)\epsilon^-. \tag{8}$$

This is a good opportunity to mention that Eq. (1) is in principle a valid equation for both fermions and bosons.

In 1992, Kähler proposed other idempotents. And this author has proposed (see his papers in arXiv) a large variety of closely related geometric idempotents based on a structure of Clifford valued differential forms. The combinations of these features leads one to anticipate that each idempotent is an attractor for solutions of differential systems. Idempotent solutions will emerge whenever there is opportunity. Clearly this concept of attractor is more general than in the present mathematical literature, but the idea is the same in the etymological sense of the term. We now show how anticipation may help bring about additional, unsuspected physical unification.

Of impredicativity, attractors and unification of conceptual spaces.

In order to reach the "individuum" (i.e. the specificity of the system or theory that one wants to obtain from the mother equation, $\partial u = u$), we write u as u_1u_2, where u_1 is the input for the specific system and u_2 is the output. In spite of the practical complexities involved in dealing with the differentiations and the products involved, it is possible to transform the equation $\partial(u_1u_2) = u_1u_2$ into $\partial u_2 = au_2$, i.e. into the first of equations (3), with a determined by u_1.

One may ask, why start with $\partial u = u$ and $u = u_1u_2$, rather than directly with $\partial u_2 = au_2$. The answer to that is that it may be easier (for being more directly connected with the nature of the problem) to try to "read' which u_1 rather than which a is at work. This mechanism seems to be custom made for statistical physics, u_2 representing the subsystem and u_1 its complement to make an isolated system. In other words, u_2 would be an "attractor solution" for the attractor input u_1.

Something more is needed, namely new variables, like the thermodynamic ones in statistical physics. As is well known, they are the response to the practical impossibility of building with any precision the trajectories of particles in a system containing more than just several of them. One achieves a rather precise evolution of mean values of a few magnitudes of the subsystem when a large number of particles is involved.

One would have to first view $\partial u = u$ with a still more general perspective. It is clear what d is in terms of whatever variables are used. This is not the case with ∂ and δ (recall $\partial = d + \delta$), but these two operations are not always needed. Also, the fact that we use new variables, like energy and volume, does not take space and time out of the picture.

In classical mechanics, we also have two types of variables: the spatial and time variables, on the one hand, and, on the other hand, the variables (α) at the root of variational principles. We then have $d = d_x + d_t + d_\alpha$. And, after removing the electromagnetic and gravitational interactions for not being of the essence of classical mechanics, what remains as input in the classical mechanics (metatheory) is the constraints. These act as the complementary subsystem. We have not yet fully developed most of the foregoing ideas. We have mentioned them as examples of how believing in daring anticipations and having the appropriate mathematical resources may give birth to the theory that leads to the anticipated result.

Anticipation and the weirdness of the Kähler calculus

When I first encountered the Kähler calculus, I found it weird, as you probably have also found. I, however, saw the great opportunities it contained as it was the basis for formulating quantum mechanics with the same language as differential geometry and general relativity. And it was giving great results, like positrons with ab initio positive energies, not to mention the fact that Kähler's continuity equation calls into question the significance of the Copenhagen interpretation. Anticipating on the basis of such considerations, among other, that Kähler's approach to quantum physics is the way to go, one also anticipates that there must be a solution to the weirdness.

To be specific, the weirdness starts with the fact that the dot product of two differential $1-$forms is a scalar in the Kähler calculus of scalar-valued differential forms, but not when they are Clifford valued. The role that one ascribes to the differential of the coordinates should be ascribed to the translation element, $d\mathbf{P}$, though it must be viewed as a vector-valued differential $1-$form, not as a tiny vector of sorts.

In the development of the aforementioned *anticipation*, *impredicativity* has come to the fore. In our dealing with these terms, we have landed on issues like external inputs, attractors and mother equations. The next step is for us, mathematicians and theoretical physicists, to deliver and to advance the pair of unifications we have spoken about using anticipations and impredicativity. The seeds for these ideas on unifications were present in my mind for many years, but they have germinated in the last year thanks in large part to exposure to these concepts provided by Professor Poli. That is why I am keen to promote his keynote lecture. The place to report on the progress just stated will be in my presently incomplete paper on Kähler's quantum mechanics, posted in this web site, where I shall keep adding sections.

To conclude, let me return to my long, opening citation. Any progress however small that one could make in developing Kähler's proposal for a universal equation, or a common language and connection of the formats of classical and quantum physics, or the unification of the interactions would certainly contribute significantly to moving the culture away from too much specialization.

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