KÄHLER´S QUANTUM MECHANICS

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ABSTRACT

This is a paper which is being posted even if incomplete in order to motivate participation in the 2017 Alterman Summer School on Clifford algebra and the Kähler calculus. The first main point we make is that the quantum theory that is a concomitant of this calculus is inconsistent with the Copenhagen interpretation of quantum mechanics.

We follow with the concept of particles in the context of the "Kähler equation", which, in first approximation, plays the same role as the Dirac equation plays in relativistic quantum mechanics. We show the advantages of the former over the latter, since it is about fields in a rather classical sense, not as probability amplitudes, which is an emerging concept under the right circumstances. Spinorial solutions, both fermionic and bosonic, emerge as particular types of "field solutions".

We then proceed with first inroads into superconductivity theory. Unlike the preceding subjects, which are not new in the Kähler literature, superconductivity has not being addressed with this calculus. For this reason, we start to be more detailed at this point, actually more than we intended for the purpose of achieveing a certain degree of consistency. We do not feel we should announce now how this paper will continue, since the 2017 Alterman event is just weeks away. Any continuation would take place later than that and without a completion target.

1 Introduction

This paper is intended to give a bird's eye view of the extraordinary power of the Kähler calculus (KC), first published in 1960-62 [1], [2], [3]. I shall concentrate on its use in KC in quantum mechanics, where its impact is greater.

Readers who do not know the mathematics might ab initio consider preposterous some of the ideas advanced here. Their interest should first be

motivated, and their minds opened. This is why this paper is what it is. But do not underestimate the mathematical quality of the arguments. Just check them in the original Kähler papers, or look at the chapters that I wrote for the 2016 Alterman event, which are reproduced in this website. Reading only the foldface, you can get a glimpse of the first two sections in less than five minutes.

Consider the following interesting circumstance. As the notable mathematician Bourguignon [5] has pointed out, a Kähler paper of 1933 [4] for which he is very famous (key words: Kähler metrics and Kähler manifolds) not only constitutes important work in differential geometry, but it also has impacted complex analysis, algebraic geometry, global analysis and theoretical physics. Somebody with a forensic investigator's mind might ask: if this work is so important, why did Kähler not return to it?

This issue has to be seen in the context that, in the last decade of his long life, he, however, returned to the KC. His 1992 paper, called "Space-time Individual" (Raum-Zeit Individuum) has a very ambitious goal [6]. Kähler tried to reach individual in systems that obey the same universal equation that he proposed. I am not advocating his ways in this case. I am simply reporting because it suggests that he might have considered his calculus one of his most important contributions to mathematics, perhaps the most important one. Old work to which authors of such a stature return at a very old age often speaks of what they have viewed as a most important part of their work.

I have taken the liberty of indicating the direction in which one could extend his efforts, both in physics and mathematics. And, if you do not like the interpretations of the mathematics (mostly by Kähler, exceptionally by I), you are invited to develop your own.

2 The foundations of Kähler's Quantum Mechanics

2.1 The Kähler equation.

Kähler's quantum mechanics is an extensive application of an elegant equation of the KC that generalizes $y' = f(x)y$. It is called the Kähler equation

(KEq) and. For scalar-valued differential forms, it reads

$$\partial u = au, \tag{1}$$

where ∂ is a Dirac-type derivative operator and where juxtaposition of the differential forms a and u (respectively input and output in the equation) means their Clifford product. **The "wave function", u, is in principle an element of the Clifford algebra of scalar-valued differential forms and not** necessarily, much less primordially, a member of an ideal in this algebra, i.e. **a spinor** (These are very, very important in Kähler's work, but in appropriate context). If one lets the math speak, quantum mechanics jumps in front of our own eyes, without mysticism.

The Clifford algebra that underlies the KC is not a tangent Clifford algebra, i.e. of the usual type, but of differential forms understood as functions of hypersurfaces. It is built from differential $1-$forms, thus upon a module rather than upon a Euclidean or pseudo-Euclidean vector space. It is defined by the equation

$$dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2g^{\mu\nu}, \tag{2}$$

which is a disguised form of the "dot product",

$$dx^\mu \cdot dx^\nu = g^{\mu\nu}. \tag{3}$$

The relation between the last two equations will be obvious to practitioners of Clifford algebra, which is the main audience that we target with this essay.

Whether expert or not, you may find this equation kind of strange. YES!, it is. Congratulations for being aware of this. I got the same impression when I first encountered it. Retrospectively, it simply happens that there is a still deeper calculus that will replace this equation with something less strange while at the same time more powerful and sophisticated, namely the tensor product of tangent Clifford algebra with the algebra of scalar-valued differential "clifforms" defined by equation 2. Kähler chose instead the tangent tensor algebra for the valuedness. He developed tangent tensor valuedness and scalar-valuedness in parallel. He did not do anything with the tensor-valuedness other than writing for it a very weird Dirac-like equation. We are interested here only in his development of the scalar-valuedness case. What he did with it would have been more than enough to have earned him the Nobel Prize of Physics if he had done his work in the 1920's. The results he obtained with scalar-valued differential forms are the ones in which we are mainly interested here.

Not being a physicist, Kähler barely spoke of the enormous physical implications of his work. Thus, the conservation law that he obtained from (1) through the first Green identity of his calculus **is inimical to the Copenhagen interpretation**, since it is for some magnitude which comprises two densities in the same continuity equation, one which is always positive and one which is always negative. u is not, therefore, a probability amplitude. The wave equation then tells us that electrons and positrons, and any particle for that matter, are extended particles, not points. The physicist Carver Mead has already made this case experimentally where he excels, in the laboratory. For his views on quantum mechanics see [7], [8]

Kähler's antiparticles emerge with the same sign as particles. So **there is no need for an (**almost full) **infinite sea of negative energy solutions.** And, as this author has discovered in recent months, **Cooper pairs, and superconductivity and topological phase transitions fall nicely in the structure of his calculus**. And Clifford valuedness takes us closer to unification of classical and quantum physics, as well as the zoo of elementary particles.

2.2 Constant idempotents, spinor solutions and density "matrices"

Kähler brought the solving of differential equations to a new level of sophistication by first writing them as exterior systems and then developing a method to deal with their solutions with symmetry. In response to a question I asked an expert on differential systems, he said this: the Cartan-Kähler theory is half of the theory of differential equations; the other half is everything else.

In general, solutions u are not members of ideals, much less are they particle states. The key of the relation between solutions in the algebra and solutions in ideals resides in the following. Idempotents define ideals. But if these are of a type called constant idempotents (let us denote them $Id^{\pm}$), the equation

$$\partial(uI^{\pm}) = (\partial u)I^{\pm}. \tag{4}$$

follows. We have not yet introduced here concepts needed to define constant idempotents. Suffice for the moment to assume that they indeed are such that they make equation (4) valid.

From (1) and (4), one gets

$$\partial(uI^{\pm}) = (\partial u)I^{\pm} = a(uI^{\pm}). \tag{5}$$

The last parenthesis is for emphasis. So, spinor solutions emerge from solutions in the algebra. Remark: Particles are spinors, but one still needs more structure before we can say "this spinor represents a particle". On the other hand, specific particles have to do with specific idempotents and, therefore, with specific spinors.

We shall later see **in connection with this emergence of spinor solutions** that **there is an intrinsic indeterminacy in Eq. (1)**. It will become obvious by contrast between the particle and non-particle contents of u that the last one is a background field with a rather chaotic contents. It nevertheless obeys the same basic equation as the particles that live in it, and of which we shall speak later. **The intrinsic indeterminacy in u is similar to that of the density matrix** of standard quantum mechanics, **except that it is an amplitude**, not a density. Spinors, say those that represent well defined particles or even systems of particles surrounded by a chaotic landscape. One cannot scape the *practical* indeterminacy of the background. But its coexistence with those structured solutions is consistent (to put it mildly) with the property of **Josephson** junctions to measure with greater precision that the background would seem to allow. The **junction has to be seen as an interchange between two stable solutions where each of the exchanging actors will change as if the background did not exist**, unless the latter overwhelms and destroys the spinors.

In order to emphasize that, **in the KC, concepts that we associate with particles are already present in non-particle field solutions**, we shall not speak about particles for a while. The association will be evident because the concepts are defined even for non-spinorial fields. The most striking example of this will be charge, but not in quantized form until particles "emerge". In the next subsection and section, we start to see how this can be, as one has to replace the probabilistic interpretation of the wave function with something else.

2.3 Dismissal of the Copenhagen interpretation

In Kähler's quantum mechanics, u also is an amplitude. This has to do with the form of the first Green identity. From the fact that the KEq lives in the algebra, and not exclusively in some ideal, the conservation law that pertains to the Kähler "wave function" (meaning wave differential form u) satisfies a different type of continuity equation. Translated to the vector calculus for

better understanding, it reads as

$$\frac{\partial \rho_1}{\partial t} + div\ \mathbf{j}_1 + \frac{\partial \rho_2}{\partial t} + div\ \mathbf{j}_2 = 0, \tag{6}$$

where ρ_1 is nowhere negative and ρ_2 is nowhere positive. $\mathbf{j}_1$ and $\mathbf{j}_2$ are the corresponding currents. ρ_1 and ρ_2 (similarly $\mathbf{j}_1$ and $\mathbf{j}_2$) are determined respectively by ${}^{+}u$ and ${}^{-}u$, defined by the constant idempotents $\epsilon^{\pm}$, where

$$\epsilon^{\pm} = \frac{1}{2}(1 \mp idt), \tag{7}$$

through the equation

$$u = {}^{+}u\ \epsilon^{+} + {}^{-}u\ \epsilon^{-}. \tag{8}$$

The conservation law that the continuity equation (6) embodies is totally incompatible with the Copenhagen interpretation, since there is not such a thing as negative probability densities. Of course, it is easy to imagine that there will be situations or systems where the probability interpretation works as an emergent concept, if one of the two similar halves of the right hand side of (6) vanishes, or if both happen to yield zero.

In situations where it applies, **the Copenhagen interpretation is an emergent or derived interpretation, rather than a basic tenet of the foundations of quantum physics**. It was adopted in the conceptual fog that accompanied the birth of quantum mechanics. It has become "truth" because virtually everybody repeats it, almost without question. If one expresses a different view, one does not belong, unless one is recognized to be in a position of too high authority to be criticized.

The superseding interpretation of u will emerge from its decomposition in terms of spinor solutions and on whether these are particle states, superpositions of particle states or none of these. And one should be aware of the fact that one does not need to assume that a is the electromagnetic coupling. Other couplings will simply give rise to different details in the expressions for $(\rho_1, \mathbf{j}_1)$ and $(\rho_2, \mathbf{j}_2)$ respectively in terms of ${}^{+}u$ and ${}^{-}u$.

3 From fields to particles

3.1 Particle concepts in fields without particles

The two terms on the right of (8) are valid for positrons and electrons, but also for concepts like charge, angular momentum and momentum in more

general systems.

Let a be the electromagnetic coupling, $m + eA$, where A is a differential $1-$form and where m and e are constants (other constants in the equation are irrelevant for present purposes and we ignore them here). Solutions in the ideals will now satisfy equations

$$\partial(u\epsilon^{\pm}) = (m + eA)(u\epsilon^{\pm}). \tag{9}$$

The conserved quantity to which the previous section referred should now be identified with a general concept of **charge**. We said "general" because it **is defined and can be computed for any given u, regardless of whether it represents a particle or not**. It is a matter of integrating the ρ's. And if a is the electromagnetic coupling, ρ is the density of what we know as electric charge. If there are no particles in the field (say, because there were not enough energy), it still is a valid concept, but not yet quantized.

One does not need to resort to analogy with non-relativistic quantum mechanics or to have a physical interpretation to m, e and A for developing equation 9. One still obtains the Pauli-Dirac equation and, in the next approximation, the Foldy-Wouthuysen equation. Assume now that we assign the standard interpretation to m, e and A. Assume further that we specialize (9) to $A = -(edt/r)$ and solve the equation. We get the fine structure of the hydrogen atom. Nowhere in what we have described is there a need for Pauli or Dirac matrices. Yet one obtains standard quantum-mechanical results for standard problems when one gives a physical interpretation to m, e and A . As for m itself, one only needs to assume that the m we choose is very close to the energy of solutions sought. The energy of the solutions is anticipated by that assumption. Concepts like the general momentum operator emerge (Operator theory is not necessary for development of Kähler's quantum mechanics, as these operators come embedded in the computations, each in its own way).

Spinors and Hilbert spaces are emerging concepts. And the concept of probability amplitude also is an emerging concept, rather than one belonging to the foundations of quantum physics, as we have seen. Many concepts emerge even before we need to introduce the concept of particle, as happens with the concept of charge. Some of them do from development of Eq. (1), as we have seen. Others do from development of equation (9). **The emergence of electrons and positrons from the KC will later be seen as being one and the same with the quantization of charge**.

3.2 The pairs (rest mass, charge) and (angular momentum, chirality)

Kähler theory reaches the standard conclusion of characterizing particles by their mass and spin, but in a different way from group representation theory. It is a matter of judiciously choosing idempotents and, among these, the primitive ones. **Charge is related to energy by virtue of their common relation to time translations, respectively through dt and t** (Present day physics has failed to realize this because it has failed to look at differential equations as per the Cartan-Kähler theory of exterior systems). Very interesting theory results from multiplication of $\epsilon^{\pm}$ with other constant idempotents with which they commute.

The role of $\partial/\partial t$ is now played by $\partial/\partial\phi$, not by a **replacement of r and p in r $\times$ p with operators**, which **is an unnecessary particle-related approach**. The idempotents associated with rotations are

$$\tau^{\pm} = \frac{1}{2}(1 \pm idxdy). \tag{10}$$

This may look a little bit strange because one might have expected this expression to involve $d\phi$. It is actually buried in $dxdy$. To make the argument short, let us simply mention that $dxdy$ is a constant idempotent, but $d\phi$ is not. The two signs inside the parenthesis correspond to two different chiralities in an obvious way. **Spin and chirality have to do with the dependence of the wave function on ϕ and $d\phi$ respectively**.

The $\tau^{\pm}$'s annul each other and commute with the $\epsilon^{\pm}$, which also annul each other. As a result of all of this, the idempotents $\epsilon^{\pm}\tau^{*}$, where τ^{*} stands for both $\tau^{\pm}$ and $\tau^{\mp}$, also annul each other.

We can use the four $\epsilon^{\pm}\tau^{*}$ to replace $(1, dt, dxdy, dtdxdy)$ in u, so that we get any member of the algebra as a sum of elements

$$u = {}^{+}u^{+}\ \epsilon^{+}\tau^{+} + {}^{+}u^{-}\ \epsilon^{+}\tau^{-} + {}^{-}u^{+}\ \epsilon^{-}\tau^{+} + {}^{-}u^{-}\ \epsilon^{-}\tau^{-}. \tag{11}$$

The coefficients of these idempotents can be computed from the fact that the identity

$$1 = \epsilon^{+}\tau^{+} + \epsilon^{+}\tau^{-} + \epsilon^{-}\tau^{+} + \epsilon^{-}\tau^{-}. \tag{12}$$

yields

$$u = u\epsilon^{+}\tau^{+} + u\epsilon^{+}\tau^{-} + u\epsilon^{-}\tau^{+} + u\epsilon^{-}\tau^{-} \tag{13}$$

through multiplication by u on the left. Because of mutual annulment of the idempotents, multiplication of (13) by $\epsilon^{\pm}\tau^*$ on the right yields

$$u\epsilon^{\pm}\tau^* = {}^{\pm}u^* \ \epsilon^{\pm}\tau^*. \tag{14}$$

There is a great difference between (11) and (13). The ${}^{\pm}u^*$ do not contain the $(1, dt, dxdy, dtdxdy)$. They are computed through (14), thanks to the properties of the idempotents involved.

The four products $\epsilon^{\pm}\tau^*$ are primitive idempotents. Readers who are not familiar with this concept need only know that one cannot carry further multiplying $\epsilon^{\pm}\tau^*$ with other idempotents and still get results such as (14). The reason is the impossibility (backed by a Radon-Hurwitz theorem [9] and also by trial until exhausting the options) is the lack of commutativity of $\epsilon^{\pm}\tau^*$ with additional idempotents factors.

3.3 The emergence of electrons and positrons, but not as point particles

A Dirac spinor can then be thought as the first column of a 4×4 matrix, the other columns then being made of zeros. The Dirac spinor accommodates electrons and positrons, both left handed and right handed, respectively represented by its four complex entries. Positrons emerge with negative energies. In contrast, these four particles would be represented by the four different columns of matrices in the Kähler calculus, if such matrix representations were used. But matrix representations are not needed (column matrices in particular) and simply create confusion.

In Kähler's quantum physics emphasizes the Clifford algebra in which physical spinors are mebers of its ideals. Kähler clearly saw this when he proposed the formulas

$$\epsilon^{\pm}\tau^* \ e^{iJ\phi}e^{-iEt/\hbar} \ f(\rho, z, d\rho, dz). \tag{15}$$

for electrons and positrons with both chiralities. The factors $\epsilon^{\pm}\tau^*$ multiplying the same factor $e^{-iEt/\hbar}$ speaks of the fact that the sign of the energy need not be different. **Antiparticles have to do with the idempotent that enters the spinor in the wave function for a given particle. What makes the positron a positron is its pertaining to the left ideal defined by ϵ^+, not negative energies**. In principle, there could be negative sign for both electrons and positrons, as also is the case for J..But this is a

separate issue. The KC does not speak of why the negative values of energy are not needed, which might do with the arrow of time or whatever. But it speaks of the fact that one does not need opposite energies to have particles and antiparticles.

Notice that $\hbar$ is absent in the first phase factor. This has to do with the way in which he defined angular momentum. Of course, this J is dimensionless. We shall return to this factor in connection with a paper by Kosterlitz [10] .

If we did not have in (15) the exponential for energy and/or angular momentum, we would have dependence of f also on t and/or ϕ. In general, the four terms in (11) would depend on $f(t, \rho, \phi, z, d\rho, dz)$. The options also go in the opposite direction, namely that one or more of the four terms take easier forms, and even that a couple of them might combine.

For Mead, an electron has the property of adapting to its environment, be it a hydrogen atom or a wire [8] He claims that experiments are regularly performed with neutrons that are one foot across. In his laboratory, he can make electrons that are ten feet long. He makes statements like "The electron ... is the thing that is wiggling, and the wave is the electron". His use of the term wave is not the standard one of classical electrodynamics (He would use the term non-coherent rather than classical). He simply means a solution of a quantum mechanical wave equation. This remark about his use of electrons as waves is crucial in order not to misunderstand him in what follows. Mead argues that the experimental support for the point particle view was due to technique that was too primitive when such a concept was adopted. In accordance with this view the factor. **In accordance with this view,** $f(\rho, z, d\rho, dz)$ **represents the shape that the extended electron or positron takes as a function of what confines it, of what it is attached to.**

To complete the picture, let us add this. Asked what should we think of a photon, Mead had this to say: "*John Cramer at the University of Washington was one of the first to describe it as a transaction between two atoms*". Then he was asked: "So that transaction is itself a wave?" Response: "The field that describes that transaction is a wave, that is right". The photon thus is the field that is being transacted.

4 Superconductivity and related issues

The purpose of this section is two fold. In the first subsections, the KC will be providing a natural perspective on features of superconductivity theory and phase transitions. In the latest ones, we make the point that concepts and equations that we shall be proposing in our development of Kähler's Quantum Physics are not far fetched or improbable, since one already finds similar equations in superconductivity theory. The five specific topics that we shall consider give title to the five subsections that follow.

4.1 Cooper pairs, idempotents and particles

Cooper pairs provide illustration of the fact that Kähler equations, including the specific ones for electromagnetic coupling, are as much for bosons as they are for fermions. In order not to complicate matters unnecessarily, we shall deal only with bosons that are composites of electrons, not those to which the paradigm assigns the role of carrying the interactions.

In the incorrect statement that the wave function is a probability amplitude, the problem is not with amplitude but with probability. As Carver Mead states in pages 68-75 of [8]: "The electron is not something that has a *fixed physical shape*" (emphasis added). And, to the question of how big an electron is, he went on to state that occasion that "It expands to fit the container it's in" (Recall that Mead considers, for instance, the H atom as a container of the electron).

Because of lack of obvious alternatives for interpretation of the factor $f(\rho, z, d\rho, dz)$ in the formula (15) for electrons and positrons of both chiralities, it is expected to provide the specifics about the state of those leptons, after one has processed the wave function through the continuity equation.

Returning to Cooper pairs, one should not think of them as a pair of electrons coming together in some way explainable only through highly complicated theory. This is certainly the way in which they emerge, but it says nothing about their mathematical representation. One should not put too much emphasis on "pair". The term Cooper particle would be more appropriate. If you insist on using "pair", it should be considered as the *interactive* "superposition" of two solutions of the just displayed types, specifically $\epsilon^{-}\tau^{\pm}$. But, by virtue of the interaction, what was a pair has become a solution of type

$$\epsilon^- e^{iJ\phi} e^{-iEt/\hbar}\ p(\rho, z, d\rho, dz), \tag{16}$$

of the equation

$$\partial p - \ (\frac{E}{\hbar c} + \beta) \vee \eta p \ - \ \alpha \vee p = 0, \tag{17}$$

where E is the mass of the Cooper pair. Without explicitly mentioning Cooper pairs, Kähler wrote the equation (17), regardless of whether one is dealing with a superconducting state or not [3]. We proceed to explain the motivation for (16) and how one proceeds to derive (17). This equation is one asking for an application, which Cooper particles provide.

Recall Eq. (13) giving the decomposition of the field's wave function as a sum of terms belonging to respective four ideals. If u is a solution of a Kähler equation, the four terms on the right of (13) also are solutions. This does not yet make them particles, electrons and positrons in particular.

By virtue of

$$\epsilon^{\pm}\tau^+ + \epsilon^{\pm}\tau^- = 1, \tag{18}$$

we might view u as

$$u = \ ^+u\ \epsilon^+ + \ ^-u\ \epsilon^-, \tag{19}$$

which is a direct consequence of (13). But his can also be written as

$$u = u\epsilon^+\tau^+ + \ u\epsilon^+\tau^- + \ ^-u\ \epsilon^- \tag{20}$$

(Recall that $\tau^+ + \tau^- = 1$). By mechanisms such as this one, there is a very large realm of possibilities for decompositions involving a great number of relevant idempotents. But, for $^-u\ \epsilon^-$ to be speciaifcally a Cooper particle it has to be of the form (16).

As for equation (17), it results from replacing (16) for u in Kähler's equation, where we can always write the coupling as

$$a = \alpha + \beta ict, \tag{21}$$

where α and β are uniquely determined by the requirement that they do not contain dt.

Cooper pairs do not have chirality. Their spin will in principle be 0 or 1. Whether one or the other, it is for the experts to decide. Of course, Kähler's quantum mechanics can in principle decide about this issue as part

of a theory of superconductivity based on the KC. Another issue is that, as is well known, Cooper pairs also exist at temperatures above the critical one, which should emerge from computations with temperature dependent inputs α and β. This also is an issue for a theory of superconductivity based on Kähler's quantum mechanics.

The existence of Cooper pairs constitutes circumstantial evidence for the association of elementary particles with spinor solutions, the type of particle being associated with the type of idempotent. That is precisely where the most promising consequence of the Kähler calculus lies. There are idempotents to nicely fit leptons, quarks, etc. See papers by this author in arXiv (type Jose G Vargas in the upper hand side corner of the main page of arxiv.org). This author anticipates more refined versions of those papers in a not too distant future.

4.2 Geometric update of spinorial solutions of Kähler type equations

The Hamiltonian for the Heisenberg model in a $2D$ system with absence of an external field can be written as

$$H_0 = -J \sum_{<ij} \mathbf{s}_i \cdot \mathbf{s}_j = -J \sum_{<ij>} \cos(\phi_i - \phi_j) \tag{22}$$

where J>0 and the sum $< ij >$ over the lattice sum is for nearest neighbors only. At other times, it is written as

$$H_0 = -J \sum_{<ij>} \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j. \tag{23}$$

We have used notation $\boldsymbol{\sigma}$ instead of the equivalent notation $\mathbf{s}$ because the former is used to think of $\boldsymbol{\sigma}$ as a linear combination of Pauli matrices. But products of matrices are Clifford products, not dot products. Hence $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j$ would not coincide with $\mathbf{s}_i \cdot \mathbf{s}_j$, but with $\mathbf{s}_i \vee \mathbf{s}_j$ $(= \mathbf{s}_i \wedge \mathbf{s}_j + \mathbf{s}_i \cdot \mathbf{s}_j)$. It is certainly correct to say that $\mathbf{s}_i \cdot \mathbf{s}_j$ equals $\cos(\phi_i - \phi_j)$, but not $\mathbf{s}_i \vee \mathbf{s}_j$, or, if you will, $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j$.

The use of $\boldsymbol{\sigma}_i$ matrices gives the incorrect impression that computations involving them are quantum mechanical. But the Pauli matrices are simply matrix representations of the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Any other interpretation is unnecessary, often misleading and utterly wrong. When one takes that into

account, computations with Heisenberg Hamiltonians look classical. The clearest indication of this is the irrelevance for their development of whether they refer to systems of spin vectors or of any other vectors which may not be directly connected with quantum mechanics. The absence of $\hbar$ in them also points in that direction. But the reason why H_0 is qunatum mechanical in spite of that absence of $\hbar$ is that it is absent in the rotational phase factor in (15).

The understanding of rotations that one needs here is only found in books on Clifford algebra. A rotation from one vector to another need only involve the plane that they together determine. Through the use of that algebra, the formula for a rotation is the same in any dimension, be it 2, 3 or whatever. It is given by a exponential in whose exponent we find the bivector representing that plane. In dimension three, but only in this dimension, one can have a direction associated with a rotation, namely the one perpendicular to that plane. But the representation by a vector no longer works in dimension $n > 3$ since there is a subspace of dimension $n - 2$ perpendicular to the (parallel) plane(s) where the rotation takes place. No axis of rotation is involved.

In $3D$, one can speak of spin vectors. But one should be careful with the use of this term and of *plane of the rotation*, as planes are represented by monomial bivectors, but also, in $3D$, by vectors perpendicular to them. It makes a difference whether there is a vector or a bivector in an exponent since $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$ but $(\mathbf{i} \wedge \mathbf{j})^2 = (\mathbf{j} \wedge \mathbf{k})^2 = (\mathbf{k} \wedge \mathbf{i})^2 = -1$. In cases like this, the presence of the unit imaginary in the exponent has the purpose of compensating for the use ob a vector where we should use a bivector.

The angular momentum of a particle or anything rotating in the plane xy, if represented as a vector in $3D$, goes in the direction of $\mathbf{k}$ (provided that one does not think of spin as something mysterious to which common sense ideas do not apply). The vectors associated with members of a collection of objects rotating in the same plane would be parallel, and the angles formed by each pair would be zero. So, we would have $\mathbf{s}_i \cdot \mathbf{s}_j = 0$ for all pairs. When researchers speak of the plane of the rotation in dealing with $2D$ systems, they simply mean that one can take a spin to coincide with another one in the plane of the system by translating and then rotating it. We have made this trivial clarification to avoid misunderstandings when, in the following, we speak of the plane of the rotation.

The formula for the rotation of anything (call it $\mathbf{v}$) that is tangent valued is given by

$$\mathbf{v}' = e^{-(1/2)\mathbf{B}\phi}\mathbf{v}e^{(1/2)\mathbf{B}\phi}, \tag{24}$$

where ϕ and $\mathbf{B}$ are the angle and the unit bivector of the plane of the rotation. Since Kähler wave functions for electrons and positrons, Eq. (15), are scalar valued, it would seem that (24) has nothing to do with them. However, as Clifford algebra shows, the imaginary numbers in exponentials related to rotations should be replaced with monomial bivectors of square -1. Kähler's $\tau^{\pm}$ and $e^{im\phi}$ should be replaced with

$$\boldsymbol{\tau}^{\pm} = \frac{1}{2}(1 \pm \mathbf{a}_1\mathbf{a}_2 dxdy), \tag{25}$$

and $e^{m\mathbf{a}_1\mathbf{a}_2\phi}$.In subsection 4.5, we shall deal with the algebra where $\boldsymbol{\tau}^{\pm}$ live.

In a true, strict $2D$ system, rotations only take place in the plane of the system, since there is no other plane available for it. All "spin vectors" (to use standard terminology even if not quite appropriate) can certainly be in the xy plane provided that whatever is spinning in the spinor takes place in planes perpendicular to it. So, (15) should be replaced with

$$f(\rho, z, d\rho, dz)e^{-\frac{1}{2}m(\mathbf{a}_{\shortparallel}\wedge\mathbf{a}_3)\phi}\ e^{-\mathbf{a}_0 Et/\hbar}\boldsymbol{\epsilon}^{+}\boldsymbol{\tau}^{\pm}. \tag{26}$$

where $\mathbf{a}_{\shortparallel}$ stands for a unit vector in the xy plane of the system of spins, and which varies from ion to ion represented by those spinors. The unit vector $\mathbf{a}_0$ is of square -1. (In Kähler's work, the signature 2 for spacetime is of the essence). The energy phase factor goes to $e^{-\mathbf{a}_0 Et}$, and $\epsilon^{\pm}$ goes to $\frac{1}{2}(1 \mp \mathbf{a}_0 dt)$. Clearly

$$e^{-\frac{1}{2}m(\mathbf{a}_{\shortparallel}\wedge\mathbf{a}_3)\phi} = \cos m(\phi/2) - \mathbf{a}_{\shortparallel}\wedge\mathbf{a}_3 \sin m(\phi/2), \tag{27}$$

and similarly for $e^{-\mathbf{a}_0 Et}$.

In (24), ϕ is the angle of a rotation. In (15) and (26), ϕ is not an angle but the azimuthal coordinate in the plane perpendicular to the spin vector. The $-\frac{1}{2}$ factor is inconsequential for present purposes, but we have introduced it in order to later connect specific ϕ angles, say in (22), with the ϕ coordinate of (26).

We finally deal with how inhomogeneous differential forms have to be understood. An example should suffice. $\lambda dxdy$ is a differential form whose integration on the surface xy is its integral, and whose evaluation on anything else is zero. The product of a function of two surfaces may be a function of a different surface as exemplified by the equation $(dxdy)(dydz) = dxdz$.

4.3 Perspective from the Kähler calculus on a Kosterlitz computation of the Hamiltonian for a vortices-endowed 2D lattice of spins

Let us assume that the $xy-$model actually is a $2D$ lattice of ions. Their spins will not be represented by (15) or (26), but the structure of their "wave function" will be very much like it except for details, however important these may be. Under this assumed similarity, we shall show the nature of the argument relating (15) to something akin to (22), and then go beyond that. We shall be focussed on the contribution of the vortices, though this will not be obvious without a full immersion in the Kosterlitz paper [10].

Another quote from the interview done to Carver Mead [8] is relevant here:

> "...Bohr put his foot on the wrong stone, the Newtonian side rather than the quantum side. The underlying reason is that Newtonian physics was phrased in terms like position and momentum and force which are all characteristics of particles. Bohr was wedded to particles." (In page 73).

In contrast, Kähler was wedded to fields. In the paradigm, angular momentum enters quantum mechanics by replacing operators for $\mathbf{r}$ and $\mathbf{p}$ in $\mathbf{r} \times \mathbf{p}$. In Kähler's quantum mechanics, on the other hand, the angular momentum operator is $\partial/\partial\phi$. Unlike $\mathbf{r} \times \mathbf{p}$ and its quantum mechanical correspondent, $\partial/\partial\phi$ is of dimension one, which is the reason why $\hbar$ does not appear in (26).

Kähler developed a concept of scalar product that generalizes the bracket. He went on to obtain Green identities and conservation laws for his calculus. It appears that he overlooked introducing in the definition of that concept the complex conjugation of one of the factors, which we know to be of the essence. And he did not discuss $< \psi \restriction O \restriction \varphi >$, where O is some operator. But these details are easily incorporated into his quantum mechanics. With these considerations in mind, we proceed to discuss an apparently classical computation by 2016 Nobel Prize Winner J. M. Kosterlitz of the explicit Hamiltonian for a configuration with $2n$ vortices of a two-dimensional "system of spins confined to rotate in the plane of a lattice". It is related to a type of superconductivity called topological.

In their report on the work by the winners of the 2016 Physics Nobel Prize [11], the members of its committee speak of "*universality class*, which can

encompass many different systems that all behave in a similar way close to the phase transition" (emphasis in original). Three pages earlier in the same report, we find statements that give a qualitative view of that universality for two dimensional systems:

> "There is a striking similarity between a two dimensional magnet and a superconducting or superfluid film. The magnetization is a vector that normally can point in any direction but in certain magnets, the spins are constrained to lie in a plane, say the xy plane, where they are free to rotate. In such an "easy plane" magnet the direction of the magnetization is determined by a single angle, θ denoting the rotation around the z-axis."

On account of these considerations, we have the following conceptual chain

From the Kähler form of the bracket $< u \mid H \mid u >$ *for spinorial solutions* u, obtain the form of the so called *spin-spin interaction term in the Heisenberg Hamiltonian for xy systems* $\Rrightarrow$ *Bring in Kosterlitz'es classical computation in a* $2D$ *lattice of spins of the specialized Hamiltonian for a configuration with* $2n$ *vortices* $\Rrightarrow$ *Realize the disguised presence of the Kähler calculus in that computation* $\Rrightarrow$ *Use universality to anticipate (infer, expect, hypothesize) that the existence of topological phase transition in a* $2D$ *superconducting material will also have to do with the form of spinorial solutions in the Kähler calculus* $\Rrightarrow$ *Given that superconductivity is an unequivocal quantum mechanical behavior, realize the role that the absence of* $\hbar$ *in the rotational phase factor of spinorial solutions of the Kähler equation has in causing some quantum mechanical computations to become classical looking at some point in the argument.*

In Kähler's case as in Dirac's, $\hbar$ is hidden in the spins. But nowhere in the standard treatment, like Kosterlitz'es is, does one derive (22) from quantum mechanics nor does one use at any point that $\mathbf{s}$ is spin. Let us give a little bit of the flavor of his computation (We shall avoid mathematical details when we think we can do without them and still convey the gist of what is going on). For convenience, Kosterlitz uses a continuum notation and proceeds to perform an expansion of the Hamiltonian (22) around a local minimum E_0. Up to quadratic terms in $\phi_i - \phi_j$, he has

$$H = H_0 - E_0 = \frac{1}{2} J \sum_{<ij>} \cos(\phi_i - \phi_j) = J \int d^2\mathbf{r} [\bigtriangledown \phi(\mathbf{r})]^2. \qquad (28)$$

He then proceeds to deal with the vortices of the system, which are specific integrals on closed curves that involves the differential of $\phi(\mathbf{r})$ .He justifies the writing of $\phi(\mathbf{r})$ as

$$\phi(\mathbf{r}) = \psi(\mathbf{r}) + \bar{\phi}(\mathbf{r}), \tag{29}$$

"where $\bar{\phi}(\mathbf{r})$ defines the angular distribution in the configuration of the local minimum of H and $\psi(\mathbf{r})$ defines the deviations from this".

He further justifies introducing the conjugate function $\bar{\phi}'(\mathbf{r})$ defined by

$$f(z) = \bar{\phi}(\mathbf{r}) + i\bar{\phi}'(\mathbf{r}), \tag{30}$$

where $z = x + iy$. Here his computation requires statements on complex variable theory that are typically unfamiliar to non-experts (One would expect that in order to define $\bar{\phi}'$ one should have previously defined $f(z)$, which is nowhere present). But nothing like that is needed since the expression $f(z)$ already involves a restriction in the process, namely the absence of z conjugate.

He invokes the Cauchy-Riemann conditions and goes through the equation

$$\bigtriangledown^2\bar{\phi}'(\mathbf{r}) = -2\pi\sum_i q_i\delta(\mathbf{r}-\mathbf{r}_i), \tag{31}$$

where the sum is the distribution function of the vortices and q_i is the strength of the vortex centered at $\mathbf{r}_i$. The solution is given in terms of a Green function that constitutes an approximation which he borrows from the theory of random walk. Notice that he directly computes $\bar{\phi}'(\mathbf{r})$, not $\bar{\phi}(\mathbf{r})$, which is what he needs for substitution in (28). But he shows that

$$\int d^2\mathbf{r}[\bigtriangledown\phi(\mathbf{r})]^2 = \int d^2\mathbf{r}[\bigtriangledown\phi'(\mathbf{r})]^2. \tag{32}$$

He thus obtains the Hamiltonian he sought, in which we are not here interested per se, since it would be an unnecessary distraction. Again, we are simply interested in discussing isues relating to the foregoing process from the perspective of Kähler's version of quantum mechanics.

We have to perform the computation $< u \mid \mathbf{a}_0\hbar\frac{\partial v}{\partial t} >$ with u and v given by (26). We proceed to give sone idea of the bracket in the Kähler calculus. For scalar-valued differential forms, he defines it as

$$(u, v) = (\zeta u \vee v) \wedge z = (\zeta u \vee v)_0 z, \tag{33}$$

where ζ is the operator that reverses the order of all products of differential $1-$forms, where z is the unit differential form of the highest grade in the algebra, and where the subscript zero chooses the zero grade part of the contents of the parenthesis. For greater correspondence with standard quantum mechanics, we should have $\zeta\bar{u}\vee v$ instead of $\zeta u\vee v$ where overbar stands for complex conjugate. This should not affect in an essential way computations in his papers where he has used (33), since $(\zeta\bar{u}\vee v)_0$ would be equivalent to starting them with $(\bar{u},v)$. But we mention this in anticipation of the fact that, later on, the imaginary units will become elements of the tangent algebra, and we shall need to know how the bracket product has to be adapted to such more general situations. To be more precise, we are about to deal with Clifford-valued differential forms, which are members of the tensor product of two Clifford algebras. In the bracket product, the operator ζ is to be understood as reversing also the order of the vectors involved in the elements of the tangent Clifford algebra. In this way, if we replace the imaginary unit with a vector or multivector of square minus one in phase factors, we obtain an $n-$form, which is an $n-$volume integrand.

An example should clarify this:

$$(e^{\lambda(\mathbf{a}_1\mathbf{a}_2)},e^{\lambda(\mathbf{a}_1\mathbf{a}_2)})=(e^{-\lambda(\mathbf{a}_1\mathbf{a}_2)}\vee e^{\lambda(\mathbf{a}_1\mathbf{a}_2)})_0 z=e^{-\lambda(\mathbf{a}_1\mathbf{a}_2)+\lambda(\mathbf{a}_1\mathbf{a}_2)}z=z. \quad (34)$$

Aside from the reversion of the order of factors in $\mathbf{a}_1\mathbf{a}_2$, we have used that, when vectors are orthogonal, their exterior product coincides with their Clifford product, alternatively denoted by juxtaposition. We are also replacing ζ with $\zeta\eta$ in Kähler's definition of the bracket, so that $\zeta\eta\mathbf{a}_o$ goes to $-\mathbf{a}_o$ and we thus have $(e^{\lambda\mathbf{a}_0},e^{\lambda\mathbf{a}_0})=z$.

4.4 The foundations of classical physics unification and the further development of the Kähler calculus

Kähler used Eq. (2) to define his algebra for scalar-valued differential forms, It is an strange equation as it is of a type to which we are used when it involves tangent vectors, but not differential forms. It is obvious that such an algebra should be contained in some more general algebra involving differential forms of higher valuedness, and that (2) should be reinterpreted, or made part of a bigger story. Based on geometrical arguments, we have found that the arena of the physics should not be spacetime but a $5D$ space canonically associated with it. In such a space, one can unify classical and quantum physics, but

we must first be convinced of the need to improve the sorry state of the foundations of physics, largely motivated by poor mathematics.

As Cartan pointed out in 1924 [12], Maxwell's equations are about relations among integrals, the integrands being scalar-valued differential forms. Representations of Maxwell's equations in terms of vector or tensor fields are off the mark for dealing with fundamental if not practical issues. But, as he also pointed out, the energy-momentum relations (i.e. the Lorentz part of classical electrodynamics) involves vector-valued differential forms, which is the realm of differential geometry.

In connection with statements on energy-momentum by Feynman that we quote below, we advance that equation (27.6) reads

$$\mathbf{E} \cdot \mathbf{j} = -\frac{\partial u}{\partial t} - \boldsymbol{\nabla} \cdot \mathbf{S}. \tag{27.6}$$

This will be easily recognized as the continuity equation. And his equations (27.14) and (27.15) are

$$u = \frac{\epsilon_0}{2}\mathbf{E} \cdot \mathbf{E} + \frac{\epsilon_0 c^2}{2}\mathbf{B} \cdot \mathbf{B}. \tag{27.14}$$

and

$$\mathbf{S} = \boldsymbol{\epsilon_0 c^2 E} \times \mathbf{B}. \tag{27.14}$$

Feynman pointed out the following ([13], Book 2, Chaper 27, Section 4, first paragraph):

> "Before we take up some application of the Poynting formulas [Eqs. (27.14) and (27.15)], we would like to say that we have not really "proved" them. All we did is to find a *possible* "u" and a *possible* "$\mathbf{S}$." How do we know that by juggling the terms around some more we could not find another formula for "u" and another formula for "$\mathbf{S}$"? The new $\mathbf{S}$ and the new u would be different, but they still would satisfy Eq. (27.6). It's possible. It can be done, but the forms that have been found always involve various *derivatives* of the field ... There are, in fact, an infinite number of different possibilities for u and $\mathbf{S}$, and so far no one has thought of an experimental way to tell which one is right. People have guessed that the simplest one is probably the correct one, but we must say that we do not know for certain which is the actual location in space of the electromagnetic field energy...".

Feynman's mentioning of the "experimental way..." is specially relevant in connection with the founding of electrodynamics on an action principle, from which one obtains a canonical energy-momentum tensor [14]. But this tensor does not coincide with the tensor whose components are the u and $\mathbf{S}$ given above. In order to obtain these, one adds to the canonical tensor another specific one among the many whose integration over all of spacetime is zero. But in doing so, one changes the physics Following the previous quotation, Feynman indeed states (ibid, second paragraph):

> It is interesting that there seems no unique way of resolve the indefiniteness of the location of the field energy. It is sometimes claimed that this problem can be resolved by using the theory of gravitation in the following argument. In the theory of gravity, all energy is the source of gravitational attraction. Therefore the energy density of electricity must be located properly if we are to know in which direction the gravity force acts. As yet, however, no one has done such a delicate experiment that the precise location of the gravitational influence of electromagnetic fields could be determined."

Different distributions of energy give different gravitational effects. If we add a tensor to the canonical electromagnetic energy-momentum tensor, we are changing the distribution of energy-momentum, thus the gravitational effect and, therefore, the physics. The canonical electromagnetic field theory with the standard action is thus inconsistent with an electrodynamics where $\mathbf{S}$ and u are given by the above formulas. Nothing has happened to solve this problem since Feynman made the foregoing remarks. We have cited Feynman simply to emphasize the relevance of the remarks, but notice that they are a matter of very simple logic.

The solution to these problems lies in the geometrization of classical electrodynamics. Under respective different approaches, both Einstein [15] and Schrödinger [16] tried and failed at it, but it is possible with the appropriate geometry. It is an amazing story, barely realized. A typical approach is to find a torsion such that the equations of the autoparallels yield the equations of motion with Lorentz force, the "gravitational force" emerging jointly.

Ringermacher [17] and I [18]succeeded in this endeavor, but the result is not rigorous in Riemannian with torsion geometry. It becomes rigorous in Finslerian bundles, even if the metric is the standard Lorentzian metric.

It may look as if this were a problem of trial and error, i.e. trying one torsion after another until finding the one that works. But it need not be so. It is fascinating that any torsion in Finslerian bundles will give you the Lorentz force, obviously up to the identification of components of the torsion with components of the electromagnetic field [19]. At first sight, this looks insane. Indeed, this would mean in particular that any Riemann-cum-torsion geometry would give rise to the Lorentz force, since the latter geometry can be lifted to a Finslerian bundle, it is a particular case of Finsler geometry. i.e. can be lifted to a Finslerian bundle. But standard Riemann-cum-torsion connections do not yield the Lorentz force.

The solution to this conundrum is simply that the same symbol means different things in different bundles. This applies in particular to $\mathbf{e}_0$ and Ω^0, where Ω^0 is the temporal component of the torsion. As for the 4-velocity, it is $\mathbf{e}_0$ itself. So the autoparallels do not depend on the metric and the full connection, but only on the metric and the $\omega_0^{\ i}$ components of the connection. When one lifts a standard connection to the Finsler bundle, the components rearrange themselves. They do so in such a way that the resulting or new Ω^0 is the only component of the torsion that contributes to the equations of motion, and it does so in such a way that the Lorentz force results in the autoparallels. God may not be malicious, but he certainly likes to play jokes.

Why is all this important in connection with the development of the calculus? The torsion, contorsion, translation element, connection, curvature and Einstein's "tensor" are $r-$vector valued differential $s-$forms. They belong to an algebra that is the tensor product of two Clifford algebras (if we are not too fuzzy at this point about what we call a Clifford algebra). And it is through this product algebra that one can go one step further in the development of geometric calculus by "correcting" the weirdness of Finsler bundles. But this is for the next subsection.

To summarize, it is through differential invariants in Finsler bundles that one can securely start to unify electrodynamics and gravitation. Our main subject in this paper is Kähler's quantum mechanics. Since the electromagnetic potential enters the Kähler equation with electromagnetic coupling, quantum physics starts to also become a theory about the same differential invariants. But there are different ways to deal with them: there is one way for classical physics and another way for quantum physics. This should not be surprising since concepts like frames, torsions and curvatures are not part of quantum physics. The connection between the Einstein tensor and the wave function will have to be found in the comparison of the classical

and quantum currents. Unification should not only seek unification of the interactions, but also of the classical and quantum theories.

4.5 Algebra for further advancement of a Kähler calculus for classical and quantum physics

The foundations of quantum mechanics are awkward. And some prominent physicists have confessed that they do not understand the Dirac theory. They are being candid. They do not lightly use the term *understand*.

Kähler's quantum mechanics, on the other hand, results from postulating equation (1) and letting the mathematics do the talking without preconceived physical or philosophical ideas. ∂ is here the Kähler derivative, a and u are the input and output differential forms, and their juxtaposition stands for Clifford product.

Although this is not generally known, the relation of classical electrodynamics and Einstein's theory of gravitation to the differential invariants that define the spacetime manifold is very direct. But this is not so with quantum physics. There is no affine and Riemannian curvatures, no torsion or contorsion, etc., in quantum physics..

The base space for spacetime's Finslerian connections is a 7-dimensional manifold, with natural coordinates, $(dt, dx^i, d\tau)$. There is no room for $7D$ tangent vectors in physics. But this is no problem. One refibrates the spacetime frames over the $7D$ manifold through a reduction of its tangent bundle. This takes place through a one-to-one correspondence between spacetime tangent vectors and equivalent classes of $7D$ tangent vectors. The refibration of a standard bundle of Riemannian-cum-torsion geometry is a Finsler bundle. In this lifting, formats change, not geometric results. For instance, the 4-velocity becomes $\mathbf{e}_0$, rather than remaining a combination of the $\mathbf{e}'_\mu s$. Thus, not surprisingly, a reshuffling of components (of torsion and curvature in particular) takes place. Up to the interpretation of components (from mathematical symbols to physical terms), the Lorentz force results in all cases. One cannot escape this fact. Add to this the assumption of teleparallelism and one has the unification of general relativity and geometrized classical electrodynamics almost complete. The fact that one is dealing with $4D$ tangent vector spaces over a $7D$ manifold is awkward, but not the only thing that is awkward in differential geometry in general, as we are about to illustrate. For greater clarity we momentarily step aside from Finsler

geometry.

In a 1922 paper, Cartan showed in passing that particles play in differential geometry a role subordinated to the frames [20]. He considered a particle in front of a frame. Without touching the particle, a differential Euclidean motion of the frame still changes its coordinates. From the mathematical statement of this, he obtained the equations of structure of Euclidean space. Hence differential geometry is just a theory of moving frames, not good enough for getting inside the particles. Such going inside is achieved by embedding the theory of moving spacetime frames in the geometry of a $5D$ Kaluza-Klein (KK) space without compactification of the fifth dimension. On curves, the fifth coordinate, τ, becomes propertime. The vector $\mathbf{u}$ dual to $d\tau$ is such that its projection upon the unit vectors $\mathbf{e}_\mu$, i.e. $\mathbf{u}\cdot\mathbf{e}_\mu$, are the components of the 4-velocity. Since these components are in turn functions of just the components of the 3-velocity, this KK space is intimately related to Finsler geometry. Whether one uses one or the other of these two formalism depends on what type of problem one is addressing.

The arena of quantum physics is a $4D$ subspace of this KK space. The classical and quantum branches of physics can be viewed as respectively associated with the subspaces spanned by $(\mathbf{e}_\mu)$ and $(\mathbf{e}_i, \mathbf{u})$, equivalently with (dt, dx^i) and $(dx^i, d\tau)$. The "metric" then is the restriction

$$d\wp(\vee,\vee)d\wp = 0 \tag{35}$$

connecting both subspaces. On curves this yields $d\tau^2 = dt^2(1-v^2)$, with $c = 1$. The time dimension is not just one more dimension like the three spatial ones, as the signature makes a great difference. The fifth dimension is still more unlike the spatial dimensions. And it does not make sense to speak of frames in the whole $5D$ space and its $(\mathbf{e}_i, \mathbf{u})$ subspace, but only in the $\mathbf{e}_\mu$ subspace.

The unification of classical physics being proposed takes place through the equations of structure and thus concerns the differential invariants that define spacetime. Without being self-contained and explicit —once we have already connected torsion with electromagnetic field— let us give a little bit of the flavor of the equations that connect differential geometry with classical physics.

The structure of a differentiable manifold endowed with a metric and arbitrary Euclidean or pseudo-Euclidean connection is given by the differential forms $\omega^\mu\mathbf{e}_\mu$ and $\omega_\nu{}^\mu\mathbf{e}_\mu$, restricted to $\omega_{\nu\mu} + \omega_{\mu\nu} = 0$. One names these forms

as $d\mathbf{P}$ and $d\mathbf{e}_\nu$ respectively, even if the system of differential equations

$$d\mathbf{P} = \omega^\mu \mathbf{e}_\mu, \qquad d\mathbf{e}_\nu = \omega_\nu{}^\mu \mathbf{e}_\mu \tag{36}$$

is not integrable. This is just a matter of notation and, therefore, $d(d\mathbf{P}) = \omega^\mu \mathbf{e}_\mu$ and $d(d\mathbf{e}_\nu)$ need not be zero. Let $\mathbf{\Omega}$ (torsion) and $\mho$ (affine, or Euclidean, or pseudo-Euclidean curvature) be defined as $d(d\mathbf{P})$ and $\mathbf{e}^\nu d(d\mathbf{e}_\nu)$. Let $\alpha_\nu{}^\mu$ be the differential $1-$forms canonically determined by the metric. Define $\beta_\nu{}^\mu$ as $\omega_\nu{}^\mu - \alpha_\nu{}^\mu$. The contorsion, $\boldsymbol{\beta}$, is the bivector-valued differential $1-$form $\beta_\nu{}^\mu \, \mathbf{e}_\mu \wedge \mathbf{e}^\nu$. We have

$$\mathbf{\Omega} = d\mathbf{P}(\wedge, .)\boldsymbol{\beta}, \qquad\qquad \textit{Einstein ``tensor''} = d\mathbf{P}(\wedge, \cdot)\mho. \tag{37}$$

We could have had other types of products, like $(\vee, \vee)$, $(\vee, \wedge)$, $(\vee, \cdot)$, $(\wedge, \vee)$ etc. We thus see that the algebra of classical physics appears to be tensor product of two Clifford type algebras (our use of "Clifford type" will be explained in the future).

Classical physics is about how particles move upon the action of external fields. Quantum physics is about "what goes inside the particles". This applies not only to the electron cloud in a hydrogen atom but also to the quarks in a nucleon, and to the quarks themselves. Think of the hydrogen atom in both of these ways (As for the Bohr atom, it is neither classical nor semiclassical). Classical physics is expected to be, if not a consequence of what quantum physics is, at least related to it at a very fundamental level. We can thus expect that the first order differential invariants are the same in both types of physics. But each type will have its own set of emerging concepts as a function of the nature of the respective problems that they seek to address, and the equations appropriate to solve them. In quantum physics, the basic equation is $\partial u = au$. The differential invariants that define the manifold are buried in the operator ∂. If a is the electromagnetic coupling, those invariants are also present there. Finally, geometry also enters this equation because, for spinors, u takes forms like (29). A further involvement with superconductivity in later subsections will guide us further in understanding in a more structural way on how an equation along the lines of $\partial u = au$ brings together these different points of contact between quantum mechanics and geometry, and also between quantum and classical mechanics. To conclude, the different nature of the basic equations in the two $4D$ subspaces reflects the fact that the concepts of classical physics cannot be transported without change to quantum physics. For the moment, let us continue with the Kosterlitz computation..

4.6 Continuation of the Kosterlitz computation

At this point we have to decide what we want to do, whether to connect with the sum over pairs, which is a step backward but helps with understanding why the Heisenberg is related to basic quantum physics, or proceed along the line of Kosterlitz'es computation.

In the first of these two options, the expression $-J\sum_{<ij>}\cos(\phi_i-\phi_j)$ implicitly assumes that one is relating small regions of the plane surrounding individual spinors and represented by their spin vectors to all other small regions of space, similarly represented. This is formally the case even though the next step is to keep only near neighbors, thus pairs of spinors. But the angles ϕ_i and ϕ_j associated with those spinors belong to different planes of rotation . We thus need to consider $(u, \mathbf{a}_o\hbar\frac{\partial v}{\partial t})$, where u and v represent the respective i and j regions. The expression (26) is then justified for both u and v since these here represent individual spinors, before the sum over pairs. In other words, the coordinate ϕ in u belongs to a plane which is different to the plane for the coordinate ϕ in v.

In the second option, $(u, \mathbf{a}_o\hbar\ \partial u/\partial t)$ would not allow us to use (26) but rather

$$f(\rho,\phi,z,d\rho,d\phi,dz)\ e^{-\mathbf{a}_0 Et/\hbar}\boldsymbol{\epsilon}^+, \tag{38}$$

the dependence on ϕ not being reducible to an exponential.

The summation in $-J\sum_{<ij>}\cos(\phi_i-\phi_j)$ belongs to the first option and presupposes that one has already performed the integration of the energy density, i.e. of $(u, \mathbf{a}_o\hbar\ \partial u/\partial t)$, resulting in a sum of terms over pairs of neighbors (That would be very clear if one simply had two spins). The problem is that one has not made use of random walk theory to extract a more manageable formula, which is what the second option does. The last equation in (28),

$$\frac{1}{2}J\sum_{<ij>}\cos(\phi_i-\phi_j)=J\int d^2\mathbf{r}[\bigtriangledown\phi(\mathbf{r})]^2, \tag{35}$$

thus amounts to a reverse engineering step, motivated by the fact that the paradigm does not provide the integral(s) from which the sum would follow. In the following, it will become clear what the integral(s) should be from which the summation should follow.

After some reduction, the bracket $(u, \mathbf{a}_o\hbar\ \partial v/\partial t)$ (for a pair a regions (not

for the whole plane of spins) takes the form

$$E = J \int e^{-(1/2)m(\mathbf{a}_3\mathbf{a}_{||}]\phi} e^{(1/2)m(\mathbf{a}_{||}\mathbf{a}_3)'\phi'} \frac{1}{2}(1 \pm \mathbf{a}_3\mathbf{a}_{||} dx^3 dx^{||}) \frac{1}{2}(1 \pm \mathbf{a}_{||}\mathbf{a}_3 dx^{||} dx^3)'. \tag{36}$$

The primes indicate belonging-ness to different planes of rotation, both of them perpendicular to the xy plane. J represents the integration over ρ and factors that keep emerging in the computational process. For example, the energy phase factors cancel each other out in the process of computing the bracket, and the energy's proper value goes into the making of J. Since $\boldsymbol{\epsilon}^+$ is an idempotent (We specialize to the + superscript since the ions in the system have positive energy), its square is $\boldsymbol{\epsilon}^+$ itself. Of its two terms, only the $1/2$ term survives, since $\mathbf{a}_o dt$ cannot be combined with anything else available to make the dt disappear and contribute to $(u, v)_0$.

In performing the product of the idempotents, we have to go through

$$\begin{aligned} \mathbf{a}_3\mathbf{a}_{||}(\mathbf{a}_{||})'\mathbf{a}_3 &= \mathbf{a}_3(\mathbf{a}_{||} \cdot \mathbf{a}'_{||})\mathbf{a}_{||} + \mathbf{a}_3(\mathbf{a}_{||} \wedge \mathbf{a}'_{||})\mathbf{a}_3 = \mathbf{a}_3[\cos(\phi_i - \phi_j) + \mathbf{a}_1\mathbf{a}_2 \sin(\phi_i - \phi_j)] = \\ &= \cos(\phi_i - \phi_j) + \mathbf{a}_1\mathbf{a}_2 \sin(\phi_i - \phi_j) = e^{\mathbf{a}_1\mathbf{a}_2(\phi_i - \phi_j)}. \end{aligned} \tag{37}$$

This subsection is not yet finished, and will be followed with sections on the London and Landau-Ginsburg equations

References

[1] Kähler, E. (1960): Innerer und äusserer Differentialkalkül, Abh. Dtsch. Akad. Wiss. Berlin, Kl. Math., Phy. Tech., **4**, 1-32.

[2] Kähler, E. (1961): Die Dirac-Gleichung, Abh. Dtsch. Akad. Wiss. Berlin, Kl. Math., Phy. Tech **1**, 1-38.

[3] Kähler, E. (1962): Der innere Differentialkalkül, Rendiconti di Matematica **21**, 425-523.

[4] Kähler, E. (1933): Über einer bemerkenswerte Hermitesche Metrik, Hamburg Abh. **9**, 173-186.

[5] Bourguignon, J.P. (2003): The Unabated Vitality of Kählerian Geometry, In Mathematical Works of Erich Kähler, 737-766, Berlin. Edited by Berndt, R. and Riemenschneider, O.

[6] Kähler, E. (1992): Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **16**, 115-177.

[7] Mead, C. A. (2000): Collective Electrodynamics: Quantum Foundations of Electromagnetism, The MIT Press, Cambridge, Massachusetts.

[8] Mead, C. A. (2001): From an anonymous interview done to him and carried in the September/October issue of the American Spectator, pp 68-75.

[9] Lounesto, P. (2002): Clifford Algebras and Spinors, Cambridge University Press, Cambridge.

[10] Kosterlitz, J.M. (1974): The Critical Properties of the Two-dimensional xy Model, J. Phys. C: Solid State Phys. **7**, 1046-1060.

[11] Royal Swedish Academy of Sciences (2016): Topological Phase Transitions and Topological Phases of Matter, Class for Physics of the Royal Swedish Academy of Sciences.

[12] Cartan, É. J. (1924): Sur les variétés à connexion affine et la théorie de la relativité généralisé, Ann. École Norm. **41**, 1-25.

[13] Feynman, R. P., Leighton, R. B., and Sands, M. (1964): The Feynman Lectures on Physics, Addison-Wesley, Reading, Massachusetts (Book 2, Chapter 27, Section 4, first paragraph):

[14] Landau, L.D., Lifshitz E.M. (1975): The Classical Theory of Fields, Pergamon Press, Oxford.

[15] Einstein, A. (1930): Théorie unitaire du champ physique, Ann. Ins. Henri Poincaré, **1**, 1-24.

[16] Schrödinger, E. (1950): Space-time Structure, Cambridge University Press, Cambridge. Reprinted 1986.

[17] Ringermacher, H.I. (1994): An Electrodynamic Connection, Class. Quantum Grav. **11**, 2383-2394.

[18] Vargas, J. G. (1991): On the Geometrization of Electrodynamics**,** Found. Physics **21**, 610-647

[19] Vargas, J. G. (2007): The Foundations of Spacetime Physics in the Light of Recent Developments in Geometry, in "Physical Interpretations of Relativity Theory, Proceedings of XIII International Scientific Meeting PIRT-2007, pp 70-81.

[20] Cartan, É. J. (1922); Sur les équations de la gravitation d'Einstein, J. Math. Pures et appliquées, **1**, 141-203.